Topology and its Applications 226 (2017) 16–30

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On inverse limits with set-valued functions on graphs, dimensionally stepwise spaces and ANRs

Masatoshi Hiraki, Hisao Kato

Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan

ARTICLE INFO

Article history:
Received 5 April 2017
Accepted 19 April 2017
Available online 26 April 2017

MSC:
primary 54F15, 54F45, 54C56
secondary 54C60, 54B20

Keywords:
Continua
Inverse limits
Inverse limits with set-valued functions
Dimension
Shape
Cell-like
FAR
ANR
Dendrite

ABSTRACT

A space X is a *dimensionally stepwise space* if $\dim X < \infty$ and for any $1 \leq m \leq \dim X$ there is an open set U_m of X such that $\dim U_m = m$. In this paper, for given inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ of compacta with upper semi-continuous set-valued functions, we introduce new indexes $\tilde{I}(\{X_i, f_{i,i+1}\})$ and $\tilde{W}(\{X_i, f_{i,i+1}\})$, and by use of the indexes we investigate topological structures of inverse limits of graphs with upper semi-continuous set-valued functions. Especially, we prove the following theorems.

Theorem 0.1. *Let G be a graph and let $f : G \to 2^G$ be an upper semi-continuous function. If the inverse limit $P = \varprojlim\{G, f\}$ with the single upper semi-continuous bonding function f is a polyhedron, then P is a dimensionally stepwise space, i.e., for any natural number i with $1 \leq i \leq \dim P$, P has a free i-simplex.*

Theorem 0.2. *If $f : G \to 2^G$ is an upper semi-continuous function on a graph G such that $\dim D_1(f^{-1}) \leq 0$ and $\tilde{W}(\{G, f\}) < \infty$, then the inverse limit $\varprojlim\{G, f\}$ with the single upper semi-continuous bonding function f is a dimensionally stepwise space.*

Theorem 0.3. *Suppose that I_i ($i \in \mathbb{N}$) is a sequence of the unit interval $I = [0, 1]$ and K_i is a finite simplicial complex in $I_i \times I_{i+1}$ satisfying that for any $x \in I_{i+1}$, $(I_i \times \{x\}) \cap |K_i| \neq \emptyset$ and for any $y \in I_i$, $(\{y\} \times I_{i+1}) \cap |K_i|$ is a nonempty connected set (=a closed interval). Let $f_{i,i+1} : I_{i+1} \to 2^{I_i}$ be the surjective*

E-mail addresses: 03104hiraki@gmail.com (M. Hiraki), hkato@math.tsukuba.ac.jp (H. Kato).

http://dx.doi.org/10.1016/j.topol.2017.04.022
0166-8641/

upper semi-continuous function defined by $G(f_{i,i+1}) = |K_i|$. *Then* $\varprojlim\{I_i, f_{i,i+1}\}$ *is an AR. Moreover, if* $\dim |K_i| \leq 1$ $(i \in \mathbb{N})$ *and* $\tilde{I}(\{I_i, f_{i,i+1}\}) = 0$, *then* $\varprojlim\{I_i, f_{i,i+1}\}$ *is a dendrite.*

1. Introduction

Inverse limits with bonding maps have played very important roles in the development of topology and topological dynamical systems. In fact, every complicated compactum can be represented by an inverse limit of finite polyhedra and simple bonding maps and conversely, inverse limits with simple bonding maps are very useful to construct complicated spaces.

In 2004, Mahavier started studying inverse limits with set-valued functions as inverse limits with closed subsets of the unit square $[0, 1] \times [0, 1]$ ([13]). Since then, several topological properties of inverse limits of compacta with upper semi-continuous set-valued functions have been studied by many authors (see [1,3,6–11,13–15]). In [8,10], Ingram and Mahavier discussed several results concerning connectedness, indecomposability and dimension of such inverse limits. Also, they investigated many interesting examples of such inverse limits of set-valued functions from the unit interval $I = [0, 1]$ to I. Such examples give us important suggestions on understanding of inverse limits. Main ideas of the present paper follow from such examples. The study of such inverse limits has developed into one of rich topics of continuum theory. Note that there are many differences between the theory of inverse limits with mappings (=single valued functions) and the theory with set-valued functions. Banič, Nall and Ingram studied topological dimension of such inverse limits (see also [1,8–10,14]). It is well-known that inverse limits of sequences of single-valued continuous functions (=mappings) have dimension bounded by the dimensions of the factor spaces. In [14], Nall proved that inverse limits of sequences of upper semi-continuous set-valued functions with 0-dimensional values have dimension bounded by the dimensions of the factor spaces. In [3], Charatonik and Roe investigated trivial shape properties of such inverse limits (see also [11]). The following general problem remains open.

Problem 1.1. What compactum can be obtained as an inverse limit with a sequence of upper semi-continuous bonding functions on graphs, especially $[0, 1]$?

Especially, we have the following problem.

Problem 1.2. What compactum can be obtained as an inverse limit with a single upper semi-continuous bonding function on a graph, especially $[0, 1]$?

In [7], Illanes proved that a simple closed curve is not an inverse limit on $[0, 1]$ with a single upper-semicontinuous bonding function. In [15], Nall proved that the arc is the only finite graph that is an inverse limit on $[0, 1]$ with a single upper-semicontinuous bonding function. Also, in [14] Nall showed that an inverse limit with a single upper semi-continuous bonding function on $[0, 1]$ cannot be an n-cell ($n \geq 2$). In [11], to evaluate the dimension of the inverse limit of given inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ of compacta with upper semi-continuous set-valued functions, we defined an index $\tilde{J}(\{X_i, f_{i,i+1}\})$. In this paper, we will introduce other indexes $\tilde{I}(\{X_i, f_{i,i+1}\})$ and $\tilde{W}(\{X_i, f_{i,i+1}\})$, and by use of the indexes we investigate the topological structures of inverse limits of graphs with upper semi-continuous set-valued functions. We give some partial answers to the above two problems.

2. Definitions and notations

In this paper, we assume that all spaces are separable metric spaces and maps (=mappings) are continuous functions. We use $\dim X$ for the topological dimension of a space X. A space X is a *dimensionally stepwise space* if $\dim X < \infty$ and for any $1 \leq m \leq \dim X$ there is an open set U_m of X such that $\dim U_m = m$.