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On the structure of braid groups on complexes

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ABSTRACT

We consider the braid group $\mathbf{B}_n(X)$ on a finite simplicial complex X, which is a generalization of those on both manifolds and graphs that have been studied already by many authors. We figure out the relationships between geometric decompositions for X and their effects on the braid groups.

As applications, we give complete criteria for both the surface embeddability and planarity for X, which are the torsion-freeness of the braid group $\mathbf{B}_n(X)$ and its abelianization $H_1(\mathbf{B}_n(X))$, respectively.

1. Introduction

The braid group $\mathbf{B}_n(D^2)$ on a 2-disk D^2 was firstly introduced by E. Artin in 1920's, and Fox and Neuwirth generalized it to braid group $\mathbf{B}_n(X)$ on an arbitrary topological space X via a *configuration space*, which is defined as follows: For a compact, connected topological space X, the *ordered configuration space* $F_n(X)$ is the set of n-tuples of distinct points in X, and the orbit space $B_n(X)$ under the action of the symmetric group $\mathbf{S}_n$ on $F_n(X)$ permuting coordinates is called the *unordered configuration space* on X:

$$F_n(X) = X^n \setminus \Delta, \quad B_n(X) = F_n(X)/\mathbf{S}_n,$$

where

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$$\Delta = \{(x_1, \ldots, x_n) | x_i = x_j \text{ for some } i \neq j\} \subset X^n.$$

Let $\bar{*}_n$ and $*_n$ be basepoints for $F_n(X)$ and $B_n(X)$, respectively. Then the *pure n-braid group* $\mathbf{P}_n(X, \bar{*}_n)$ and *(full) n-braid group* $\mathbf{B}_n(X, *_n)$ are defined to be the fundamental groups of the configuration spaces $F_n(X)$ and $B_n(X)$, respectively. We will suppress basepoints and denote these groups by $\mathbf{P}_n(X)$ and $\mathbf{B}_n(X)$ unless any ambiguity occurs.

However, most of research on braid groups has been focused on braid groups on manifolds, more specifically, on surfaces, until the end of 20th century when Ghrist presented a pioneering paper [1] about braid groups on *graphs* Γ which are finite, 1-dimensional simplicial complexes. In 2000, Abrams defined in his Ph.D. thesis [2] a combinatorial version of a configuration space, called a *discrete configuration space*, consisting of n open cells in Γ having pairwise no common boundaries. A discrete configuration space has the benefit that it admits a cubical complex structure making the description of paths of points easier. However it depends not only on homeomorphic type but also the cell structure of the underlying graph Γ. Abrams overcame this problem by proving stability up to homotopy under the subdivision of edges once Γ is sufficiently subdivided.

Crisp and Wiest in [3] showed the embeddability of surface groups and graph braid groups into right-angled Artin groups. Farley and Sabalka in [4] used Forman's *Discrete Morse theory* [5] on discrete configuration spaces to provide an algorithmic way to compute a presentation of $\mathbf{B}_n(\Gamma)$, and furthermore they figured out the relation between braid groups on trees and right-angled Artin groups. On the extension of these works, Kim-Ko-Park in [6] and Ko-Park in [7] provided geometric criteria for the braid group on a given graph to be a right-angled Artin group, and moreover a new algebraic criterion for the planarity of a graph.

On the contrary, for a simplicial complex, not manifold, of dimension 2 or higher, braid theory is still unexplored. We will focus on the braid group on a finite, connected simplicial complex X of arbitrary dimension, which is generalizations of both graphs and surfaces. We consider *surgeries*—attaching or removing higher cells, edge contraction or inverses, and so on—and how these surgeries change the corresponding braid groups. Indeed, via suitable surgeries we may obtain a *simple* complex X' of dimension 2 whose vertices have very obvious links. Furthermore, this can be done without changing the corresponding braid group.

Theorem 1.1. *Let X be a complex. Then there is a simple complex X' of dimension* 2 *such that $\mathbf{B}_n(X) \simeq \mathbf{B}_n(X')$ for all $n \geq 1$.*

Once we have a simple complex X, then it can be decomposed by *cuts* into much simpler pieces, and eventually into *elementary* complexes, where an elementary complex plays the role of a building block and can be thought as either a *star* graph or a manifold of dimension at least 2. For the build-up process, we provide two types of *combination theorems* which are generalizations of capping-off and connected sum. Furthermore, the combination theorems ensure that the build-up process preserves some geometry of the given pieces. In other words, the braid group $\mathbf{B}_n(X)$ captures some geometric properties of X as observed before.

More precisely, we start with the obvious observations about the various embeddability of X into manifolds as follows: For two complexes X and Y, we denote by $Y \subset X$ and say that X *contains* Y if there is an simplicial embedding between them after sufficient subdivisions. Then we recall that a complex X embeds into (i) a circle iff $T_3 \not\subset X$; (ii) a surface iff $S_0 \not\subset X$; and (iii) a plane iff $K_5, K_{3,3}, S_0 \not\subset X$.

The complexes T_3 and S_0 are the *tripod* and the cone $C(S^1 \sqcup \{*\})$ of the union of a circle and a point, respectively. See Fig. 1. The graphs K_n and $K_{m,n}$ are complete and complete bipartite graphs, respectively.

Then it can be formulated as follows.