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Monotone insertion of semi-continuous maps to ordered topological vector spaces ☆

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ABSTRACT

In this paper, we consider the problem of inserting semi-continuous maps with values into some ordered topological vector spaces in a monotone fashion. We answer the questions of K. Yamazaki and provide a new characterization of stratifiable spaces, semi-stratifiable spaces, countably paracompact spaces, countably metacompact spaces, perfect spaces, and perfectly normal spaces during this process.

1. Introduction

People have extensively investigated a problem for a given pair of real-valued (non-continuous) functions (g, h) on a space X and $g \leq h$ ($g(x) \leq h(x)$ for each $x \in X$) under what conditions there exists a continuous function f such that $g \leq f \leq h$ since the 1920's. H. Hahn [8] first considered the particular case in which

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g is upper semi-continuous and h is lower-continuous. Many so called insertion results present some classic characterizations of topological spaces, such as normal spaces, stratifiable spaces and others.

Definition 1.1. [6] A space X is said to be monotonically countably metacompact (MCM) if there is an operator U assigning to each decreasing sequence $(D_j)_{j\in\mathbb{N}}$ of closed sets with empty intersection, a sequence of open sets $U((D_j)) = (U(n,(D_j)))_{n\in\mathbb{N}}$ such that

(1) $D_n \subseteq U(n,(D_j))$ for each $n \in \mathbb{N}$;
(2) $\bigcap_{n\in\mathbb{N}} U(n,(D_j)) = \emptyset$;
(3) given two decreasing sequences of closed sets $(F_j)_{j\in\mathbb{N}}$ and $(E_j)_{j\in\mathbb{N}}$ such that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$, then $U(n,(F_j)) \subseteq U(n,(E_j))$ for each $n \in \mathbb{N}$.

X is said to be monotonically countably paracompact (MCP) if, in addition,

(2') $\bigcap_{n\in\mathbb{N}} \overline{U(n,(D_j))} = \emptyset$.

C. Good, R. Knight and I. Stares [6] and C. Pan [11] introduced this definition respectively, and it was proved in [6, Proposition 14] that both these notions are equivalent. C. Good, R. Knight and I. Stares [6], P.F. Yan and E.G. Yang [16], L.H. Xie and P.F. Yan [12], and K. Yamazaki [13] characterized monotone countably paracompact spaces and monotone countably metacompact spaces with monotone insertions of real-valued functions. By extending the insertion properties of real-valued maps, K. Yamazaki considered the semi-continuous maps with values into the ordered topological spaces, and proposed the following open problems:

Question 1.2. [14] Let X be a topological space and Y an ordered topological vector space with a positive interior point. Are the following conditions (1) and (2) equivalent?

(1) X is monotonically countably paracompact.
(2) There exist operators Φ, Ψ assigning to each lower semi-continuous map $f : X \to Y^+\backslash\{\mathbf{0}\}$, an upper semi-continuous map $\Phi(f) : X \to Y^+\backslash\{\mathbf{0}\}$ and a lower semi-continuous map $\Psi(f) : X \to Y^+\backslash\{\mathbf{0}\}$ with $\Psi(f) \leq \Phi(f) \leq f$ such that $\Phi(f) \leq \Phi(f')$ and $\Psi(f) \leq \Psi(f')$ whenever $f \leq f'$.

Question 1.3. [14] Let X be a topological space and Y an ordered topological vector space with a positive interior point. Are the following conditions (1) and (2) are equivalent?

(1) X is monotonically countably metacompact.
(2) There exist an operator Φ assigning to each lower semi-continuous map $f : X \to Y^+ \setminus \{\mathbf{0}\}$, an upper semi-continuous map $\Phi(f) : X \to Y^+\backslash\{\mathbf{0}\}$ with $\Phi(f) \leq f$ such that $\Phi(f) \leq \Phi(f')$ whenever $f \leq f'$.

The results in [6] and [12] suggest that these insertion properties with maps to ordered topological vector spaces may be equivalent to MCP and MCM. This article is just enlightened by the thoughts. We improve the conditions and give others equivalent characterizations of MCP and MCM.

Characterizations of stratifiable spaces and semi-stratifiable spaces by monotone insertions of real-valued functions are also given in [7,10,12,16,9]. We generalize the monotone insertion properties from real topological space to some ordered topological vectors spaces. The purpose of this paper is to attempt to characterize stratifiable spaces and semi-stratifiable spaces by insertion properties of maps with values into some ordered topological vectors spaces along the same lines.

Throughout this paper, all the undefined topological concepts can be found in [4]. All spaces are assumed to be Hausdorff topological spaces.