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Nonnormality of Čech–Stone-remainders of topological groups

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It is known that every remainder of a topological group is Lindelöf or pseudocompact. Motivated by this result, we study in this paper the question when a topological group G has a normal remainder. Under mild conditions on G we show that under the Continuum Hypothesis, if the Čech–Stone remainder G^* of G is normal, then it is Lindelöf.

1. Introduction

All topological spaces under discussion are Tychonoff.

By a *remainder* of a space X we mean the subspace $bX \setminus X$ of a compactification bX of X. Among the best known remainders are the Čech–Stone remainders $X^* = \beta X \setminus X$ for arbitrary spaces X and the 1-point remainders $\alpha Y \setminus Y$ for locally compact spaces Y.

Remainders of topological groups are much more sensitive to the properties of topological groups than the remainders of topological spaces are in general. A nice example demonstrating this, is Arhangel'skii's

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Theorem from [4]: every remainder of a topological group is Lindelöf or pseudocompact. All remainders of locally compact groups are compact, hence both Lindelöf and pseudocompact. For non-locally compact groups there is a dichotomy: every remainder is either Lindelöf or pseudocompact.

Observe that if X is a separable and metrizable topological space, then it has a separable metrizable compactification. The remainder of this compactification is separable metrizable as well, and hence Lindelöf. This implies that the Čech–Stone remainder $X^* = \beta X \setminus X$ of X is Lindelöf, being a perfect preimage of a Lindelöf space. Hence all remainders of X are Lindelöf since every remainder is a continuous image of X^*. Similarly, if a space X has at least one Lindelöf remainder, then all remainders are Lindelöf. (This is folklore.)

In this paper we are interested in the question when the normality of a remainder of a topological group forces that remainder to be Lindelöf, or forces other remainders to be normal.

As we saw above, this is always the case for separable metrizable groups. But not always so, as can be demonstrated by an example that was brought to our attention by Buzyakova (for a different reason). Supply $G = \{0,1\}^{\omega_1}$ with the topology generated by all boxes that are determined by countably many coordinates. Then G is a topological group, is linearly ordered and hence has a linearly ordered compactification. Hence the remainder of G in this compactification is monotonically normal and therefore, hereditarily normal. But that remainder is not Lindelöf, simply observe that G is a P-space and that any P-space with a Lindelöf remainder is discrete. We will show that the Čech–Stone remainder of this topological group G is not normal. Hence it is not true that the normality of a specific remainder implies that all remainders are normal. Hence normal remainders behave differently compared to Lindelöf remainders.

There are many results in the literature on so called points of nonnormality in Čech–Stone remainders. A point x of a space X is said to be a point of nonnormality, if $X \setminus \{x\}$ is not normal. It was shown by Gillman (see [9]) that under CH, every non-P-point is a point of nonnormality of ω^*. This also holds for P-points, as was shown independently by Warren [23] and Rajagopalan [20]. Hence under CH, all points of ω^* are nonnormality points. The first nonnormality points in ω^* in ZFC, were constructed by Błaszczyk and Szymański [8]. The question whether every point of ω^* is a nonnormality point in ZFC remains unsolved and is a classical problem by now. More recent results on nonnormality points in Čech–Stone compactifications can be found e.g. in Bešlagić and van Douwen [7], Logunov [15], Terasawa [21], and Fleissner and Yengulalp [11].

So there is quite an extensive literature on nonnormality points in Čech–Stone remainders, and history tells us that these results and the remaining problems are complicated. It is only recently that the question of when a remainder of a topological group is normal was asked for the first time in Arhangel'skii [5] (Section 3). It was shown there that no Dowker space can be a remainder of a topological group (Theorem 3.1).

Surprisingly, we are unaware of any other question in the literature that asks for conditions on X that imply that X^* is normal, or some remainder of X is normal, whereas for Lindelöf remainders such conditions are well-known (Henriksen and Isbell [13]). We will show that for G a nowhere locally compact topological group that contains a nonempty compact G_δ-subset, if the character of G is at most $\mathfrak{c}$, and G^* is normal, then G^* is Lindelöf under CH. We will formulate several applications of this result, for example to Moscow topological groups which constitute a rather large class of topological groups.

2. Preliminaries

We abbreviate Čech–Stone remainder as *CS-remainder*.

A subspace Y of a space X is said to be C^*-*embedded* in X if every bounded continuous function $f : Y \to \mathbb{R}$ can be extended to a bounded continuous function $\bar{f} : X \to \mathbb{R}$. Here $\mathbb{R}$ denotes the space of real numbers.