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Reversible filters

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ABSTRACT

A space is reversible if every continuous bijection of the space onto itself is a homeomorphism. In this paper we study the question of which countable spaces with a unique non-isolated point are reversible. By Stone duality, these spaces correspond to closed subsets in the Čech–Stone compactification of the natural numbers $\beta\omega$. From this, the following natural problem arises: given a space X that is embeddable in $\beta\omega$, is it possible to embed X in such a way that the associated filter of neighborhoods defines a reversible (or non-reversible) space? We give the solution to this problem in some cases. It is especially interesting whether the image of the required embedding is a weak P-set.

1. Introduction

A topological space X is reversible if every time that $f : X \to X$ is a continuous bijection, then f is a homeomorphism. This class of spaces was defined in [10], where some examples of reversible spaces were given. These include compact spaces, Euclidean spaces $\mathbb{R}^n$ (by the Brouwer invariance of domain theorem) and the space $\omega \cup \{p\}$, where p is an ultrafilter, as a subset of $\beta\omega$. This last example is of interest to us.

Given a filter $\mathcal{F} \subset \mathcal{P}(\omega)$, consider the space $\xi(\mathcal{F}) = \omega \cup \{\mathcal{F}\}$, where every point of ω is isolated and every neighborhood of $\mathcal{F}$ is of the form $\{\mathcal{F}\} \cup A$ with $A \in \mathcal{F}$. Spaces of the form $\xi(\mathcal{F})$ have been studied before, for example by García-Ferreira and Uzcátegi ([6] and [7]). When $\mathcal{F}$ is the Fréchet filter, $\xi(\mathcal{F})$ is homeomorphic to a convergent sequence, which is reversible; when $\mathcal{F}$ is an ultrafilter it is easy to prove that $\xi(\mathcal{F})$ is also reversible, as mentioned above. Also, in [2, section 3], the authors of that paper study when $\xi(\mathcal{F})$ is reversible for filters $\mathcal{F}$ that extend to precisely a finite family of ultrafilters (although these results are expressed in a different language).

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Let us say that a filter $\mathcal{F} \subset \mathcal{P}(\omega)$ is reversible if the topological space $\xi(\mathcal{F})$ is reversible. It is the objective of this paper to study reversible filters. First, we give some examples of filters that are reversible and others that are non-reversible, besides the trivial ones considered above. Due to Stone duality, every filter $\mathcal{F}$ on ω gives rise to a closed subset $K_\mathcal{F} \subset \omega^* = \beta\omega \setminus \omega$ (defined below). Then our main concern is to try to find all possible topological types of $K_\mathcal{F}$ when $\mathcal{F}$ is either reversible or non-reversible. Our results are as follows.

- Given any compact space X embeddable in $\beta\omega$, there is a reversible filter $\mathcal{F}$ such that X is homeomorphic to $K_\mathcal{F}$. (Theorem 3.2)
- Given any compact, extremally disconnected space X embeddable in $\beta\omega$, there is a non-reversible filter $\mathcal{F}$ such that X is homeomorphic to $K_\mathcal{F}$. (Theorem 3.5)
- If X is a compact, extremally disconnected space that can be embedded in ω^* as a weak P-set and X has a proper clopen subspace homeomorphic to itself, then there is a non-reversible filter $\mathcal{F}$ such that X is homeomorphic to $K_\mathcal{F}$ and $K_\mathcal{F}$ is a weak P-set of ω^*. (Theorem 4.1)
- There is a compact, extremally disconnected space X that can be embedded in ω^* as a weak P-set and every time $\mathcal{F}$ is a filter with X homeomorphic to $K_\mathcal{F}$ and $K_\mathcal{F}$ is a weak P-set, then $\mathcal{F}$ is reversible. (Theorem 4.2)
- Given any compact, extremally disconnected space X that is a continuous image of ω^*, there is a reversible filter $\mathcal{F}$ such that X is homeomorphic to $K_\mathcal{F}$ and $K_\mathcal{F}$ is a weak P-set of ω^*. (Theorem 4.4)

Also, in section 5, using Martin's axiom, we improve some of the results above by constructing filters $\mathcal{F}$ such that $K_\mathcal{F}$ is a P-set.

2. Preliminaries and a characterization

Recall that $\beta\omega$ is the Stone space of all ultrafilters on ω and $\omega^* = \beta\omega \setminus \omega$ is the space of free ultrafilters. We will assume the reader's familiarity with most of the facts about $\beta\omega$ from [9]. Recall that a space is an F-space if every cozero set is C^*-embedded. Since ω^* is an F-space we obtain some interesting properties. For example, every closed subset of ω^* of type G_δ is regular closed and every countable subset of ω^* is C^*-embedded. We will also need the more general separation property.

2.1. Theorem. *([3, 3.3]) Let $\mathcal{B}$ and $\mathcal{C}$ be collections of clopen sets of ω^* such that $\mathcal{B} \cup \mathcal{C}$ is pairwise disjoint, $|\mathcal{B}| < \mathfrak{b}$ and $\mathcal{C}$ is countable. Then there exists a non-empty clopen set C such that $\bigcup\mathcal{B} \subset C$ and $(\bigcup\mathcal{C}) \cap C = \emptyset$.*

We will be considering spaces embeddable in $\beta\omega$. There is no ZFC characterization of spaces embeddable in $\beta\omega$ but we have the following embedding results. A space is extremally disconnected (ED, for short) if the closure of every open subset is open.

2.2. Theorem.

- *[9, 1.4.4] Under CH, any closed subspace of ω^* can be embedded as a nowhere dense P-set.*
- *[9, 1.4.7] Every compact ED space of weight $\leq \mathfrak{c}$ embeds in ω^*.*
- *[9, 3.5], [4] If X is an ED space and a continuous image of ω^*, then X can be embedded in ω^* as a weak P-set.*

Given $A \subset \omega$, we denote $\mathrm{cl}_{\beta\omega}(A) \cap \omega^* = A^*$. Also, if $f : \omega \to \omega$ is any bijection, there is a continuous extension $\beta f : \beta\omega \to \beta\omega$ which is a homeomorphism; denote $f^* = \beta f{\restriction}_{\omega^*}$.

The Fréchet filter is the filter $\mathcal{F}_r = \{A : \omega \setminus A \in [\omega]^{<\omega}\}$ of all cofinite subsets of ω and we will always assume that our filters extend the Fréchet filter. Each filter $\mathcal{F} \subset \mathcal{P}(\omega)$ defines a closed set $K_\mathcal{F} = \{p \in \beta\omega : \mathcal{F} \subset p\}$ that has the property that $A \in \mathcal{F}$ iff $K_\mathcal{F} \subset A^*$ and moreover, $K_\mathcal{F} = \bigcap\{A^* : A \in \mathcal{F}\}$. Notice that