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On hereditarily reversible spaces

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A B S T R A C T

In [15] Rajagopalan and Wilansky called a space reversible if each continuous bijection of the space onto itself is a homeomorphism. They called also a space hereditarily reversible if each its subspace is reversible. We characterize the hereditarily reversible spaces in several classes of topologicals spaces, in particular, in the class of Hausdorff spaces of the first countability and in some subclass of the class of locally finite T_0-spaces relevant to digital topology. Besides we suggest different examples of (non-)reversible and (non-)hereditarily reversible spaces.

1. Introduction

In [15] Rajagopalan and Wilansky called a space X *reversible* if each continuous bijection of the space onto itself is a homeomorphism, otherwise X is called *non-reversible*.

It is easy to see that the property is topological, i.e. if two topological spaces are homeomorphic and one of them is reversible then the other one is also reversible.

Simple examples of reversible spaces are discrete spaces D_κ of cardinality $\kappa \geq 0$ (set $D_0 = \emptyset$), compact Hausdorff spaces, Euclidean spaces $\mathbb{R}^n$, finite spaces.

Let us note that a space (X, τ) is non-reversible iff there exists a strictly strongly (resp. weakly) topology τ_s (resp. τ_w) on the set X such that (X, τ) is homeomorphic to (X, τ_s) (resp. (X, τ_w)).

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The topological union $N_{\aleph_0} = D_{\aleph_0} \oplus cD_{\aleph_0}$ of $D_{\aleph_0}$ and $cD_{\aleph_0}$, where $cD_{\aleph_0}$ is a one-point compactification of $D_{\aleph_0}$, is non-reversible as well as the subspace $\mathbb{Q}$ (resp. $\mathbb{P}$) of rational (resp. irrational) numbers of the real line $\mathbb{R}$.

One observes easily an irregular behavior of the reversibility under simple operations (as topological union and product). Thus the construction/identification of (non-)reversible spaces is one of the main subjects of this theory. Besides [15] many examples of both types one can find in [7,3,4,13] etc.

In [15] Rajagopalan and Wilansky considered also *hereditarily reversible* spaces. (A space is hereditarily reversible if each its subspace is reversible.) They notated that an identification of hereditarily reversible spaces appeared difficult.

Besides trivial examples of hereditarily reversible spaces as

(a) any discrete space,
(b) any space with only finitely many open sets, in particular, any finite space,
(c) any space with cofinite topology,

Rajagopalan and Wilansky suggested only 'one' example of a hereditarily reversible non-discrete Hausdorff space, namely: a subspace $V_x = \{x\} \cup \mathbb{N}$ of the Stone–Cech compactification $\beta\mathbb{N}$, where $\mathbb{N}$ is the space of natural numbers and $x \in \beta\mathbb{N} \setminus \mathbb{N}$.

In this paper we will look for hereditarily reversible spaces (both Hausdorff and non-Hausdorff) and their characterizations.

2. Hereditarily reversible Hausdorff spaces of the countable character

Let X be a space and $p \in X$. We denote by $\chi(p, X)$ the character of X at the point p and by $\chi(X)$ the character of X.

Theorem 2.1. *Let X be a infinite Hausdorff space and $\chi(X) \le \aleph_0$. If X is neither homeomorphic to D_κ for any $\kappa \ge \aleph_0$ nor $cD_{\aleph_0}$ then X contains a copy of $N_{\aleph_0}$ and hence X is not hereditarily reversible.*

Proof. Since the space X is not discrete, there is a limit point p of X, i.e. $p \in \mathrm{Cl}_X(X \setminus \{p\})$.

Case 1. For each open nbd W of p we have $|X \setminus W| < \aleph_0$.

This implies that the subspace $X \setminus \{p\}$ of X is homeomorphic to $D_{\aleph_0}$ and the space X is homeomorphic to $cD_{\aleph_0}$. We have a contradiction.

Case 2. There exists an open nbd W of p such that $|X \setminus W| \ge \aleph_0$.

Note that $\chi(p, X) = \aleph_0$. Let $\{B_n : n = 1, 2, \dots\}$ be a base at p. We can assume that $B_{n+1} \subseteq B_n$ for each $n \ge 1$. There exists B_{n_1} such that $B_{n_1} \subseteq W$. Choose a point $x_1 \in B_{n_1} \setminus \{p\}$. Since X is Hausdorff, there exist open disjoint subsets O_1 and V_1 of B_{n_1} such that $p \in O_1$ and $x_1 \in V_1$. Choose $n_2 > n_1$ such that $B_{n_2} \subseteq O_1$, and then $x_2 \in B_{n_2} \setminus \{p\}$, disjoint open subsets O_2 and V_2 of B_{n_2} such that $p \in O_2$ and $x_2 \in V_2$. Continuing as above we get a sequence $n_1 < n_2 < \dots$ of integers, a sequence of points $\{x_n\}_{n=1}^{\infty}$, two sequences $\{O_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ of open sets. It is easy to see that the subspace $Y_p^W = \{p\} \cup \{x_n : n \ge 1\}$ of X is homeomorphic to $cD_{\aleph_0}$.

Case 2.a. Each point of $X \setminus W$ is not a limit point in X.

So each point of $X \setminus W$ is an open subset of X. Hence the set $X \setminus W$ is clopen in X and the subspace $X \setminus W$ is homeomorphic to D_τ for $\tau = |X \setminus W| \ge \aleph_0$. Choose any infinite countable subset Z of $X \setminus W$. Note that the subspace $Y_p^W \cup Z$ of X is homeomorphic to $N_{\aleph_0}$.

Case 2.b. The set $X \setminus W$ contains a limit point q of X.