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Remarks on monotone (weak) Lindelöfness

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ABSTRACT

Using Erdös–Rado's theorem, we show that (1) every monotonically weakly Lindelöf space satisfies the property that every family of cardinality $\mathfrak{c}^+$ consisting of nonempty open subsets has an uncountable linked subfamily; (2) every monotonically Lindelöf space has strong caliber $(\mathfrak{c}^+, \omega_1)$, in particular a monotonically Lindelöf space is hereditarily $\mathfrak{c}$-Lindelöf and hereditarily $\mathfrak{c}$-separable. (1) gives an answer of a question posed in Bonanzinga, Cammaroto and Pansera [3], and (2) gives partial answers of questions posed in Levy and Matveev [15]. Some other properties on monotonically (weakly) Lindelöf spaces are also discussed. For example, we show that the Pixley–Roy space $PR(X)$ of a space X is monotonically Lindelöf if and only if X is countable and every finite power of X is monotonically Lindelöf.

1. Introduction

Throughout this paper, all spaces are assumed to be regular T_1. The symbol $\mathbb{D}$ is the space consisting of the two points 0 and 1. For an infinite cardinal κ, κ^+ is the successor cardinal of κ. The continuum hypothesis is denoted by CH, and $\mathfrak{c}$ is the continuum. For a space X, we denote by $w(X)$ (resp., $nw(X)$, $d(X)$, $\chi(X)$, $\psi(X)$, $s(X)$, $c(X)$) the *weight* (resp., *net-weight*, *density*, *character*, *pseudocharacter*, *spread*, *cellularity*) of X [7]. For families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a set X, we say that $\mathcal{A}$ *refines* $\mathcal{B}$ if every member of $\mathcal{A}$ is contained in some member of $\mathcal{B}$. In this paper, $\mathcal{A}$ and $\mathcal{B}$ are not assumed to be a cover of X. If $\mathcal{A}$ refines $\mathcal{B}$, we write $\mathcal{A} \prec \mathcal{B}$.

Definition 1.1. A space X is *monotonically Lindelöf* (abbr., *mL*) if there is a function r (called a *mL-operator*) that assigns to each open cover $\mathcal{U}$ of X a countable open cover $r(\mathcal{U})$ of X such that (a) $r(\mathcal{U}) \prec \mathcal{U}$ and (b) $r(\mathcal{V}) \prec r(\mathcal{U})$ for any open covers $\mathcal{V}$ and $\mathcal{U}$ of X with $\mathcal{V} \prec \mathcal{U}$.

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Every mL-space is obviously Lindelöf. This notion was first introduced by Matveev in [16], and recently studied in [2,8,13,14] and [15]. Every second countable space is mL, but neither a compact space nor a countable space is always mL. Every closed subspace of a mL-space is mL.

A space X is said to be *weakly Lindelöf* if every open cover $\mathcal{U}$ of X contains a countable subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup \mathcal{V}$ is dense in X. A monotone version of weak Lindelöfness was introduced and studied in Bonanzinga, Cammaroto and Pansera [3].

Definition 1.2 *([3, Definition 2.1]).* A space X is *monotonically weakly Lindelöf* (abbr., *mwL*) if there is a function r (called a *mwL-operator*) that assigns to each open cover $\mathcal{U}$ of X a countable open family $r(\mathcal{U})$ in X such that (a) $r(\mathcal{U}) \prec \mathcal{U}$, (b) $\bigcup r(\mathcal{U})$ is dense in X and (c) $r(\mathcal{V}) \prec r(\mathcal{U})$ for any open covers $\mathcal{V}$ and $\mathcal{U}$ of X with $\mathcal{V} \prec \mathcal{U}$.

Every mwL-space is obviously weakly Lindelöf. It is easy to see that a space with a countable π-base is mwL. Not every closed subspace of a mwL-space is mwL [3, Theorem 3.1].

In this paper, we further study monotonically (weakly) Lindelöf spaces, and give some (partial) answers of questions posed in [3] and [15].

2. Monotonically (weakly) Lindelöf spaces

For a cardinal $\kappa \geq \omega$, let $A(\kappa) = \{p\} \cup D(\kappa)$ be the one-point compactification of the discrete space $D(\kappa)$ of cardinality κ. It is shown in [13, Corollary 30] that $A(\kappa)$ is mL if and only if $\kappa = \omega$ holds. In contrast with this fact, if $\kappa \leq \mathfrak{c}$ holds, then $A(\kappa)$ is mwL [3, Theorem 2.7]. Thus, $A(\omega_1)$ is a mwL-space which is not mL. First we answer the following question.

Question 2.1 *([3, Problem 2.8]).* For what cardinals $\kappa > \mathfrak{c}$ is $A(\kappa)$ mwL?

Lemma 2.2 *(Erdös–Rado's theorem, [12, p. 290]). Let I be a set with $|I| = \mathfrak{c}^+$, and let $[I]^2 = \{s \subset I : |s| = 2\}$. Then, for any function $f : [I]^2 \to \omega$, there are a subset $H \subset I$ and a $k \in \omega$ such that $|H| = \omega_1$ and $f(s) = k$ for any $s \in [H]^2$.*

A family $\mathcal{L}$ of sets is said to be *linked* if $L \cap L' \neq \emptyset$ for any $L, L' \in \mathcal{L}$. A space X is said to have *property* $K(\mathfrak{c}^+, \omega_1)$ if every family of cardinality $\mathfrak{c}^+$ consisting of nonempty open subsets in X has an uncountable linked subfamily.

Theorem 2.3. *Every mwL-space has property $K(\mathfrak{c}^+, \omega_1)$.*

Proof. Let X be a mwL-space, and let r be a mwL-operator for X. Assume that there is a family $\{D_\alpha : \alpha < \mathfrak{c}^+\}$ consisting of nonempty open subsets in X such that any uncountable subfamily is not linked. For each $\alpha < \mathfrak{c}^+$, we take a nonempty open set E_α in X with $\overline{E}_\alpha \subset D_\alpha$. For each $\alpha < \mathfrak{c}^+$, consider the open cover $\mathcal{U}_\alpha = \{D_\alpha, X \setminus \overline{E}_\alpha\}$ of X, and let $r(\mathcal{U}_\alpha) = \{V_{\alpha,n} : n \in \omega\}$. We put $[\mathfrak{c}^+]^2 = \{(\alpha, \beta) : \alpha < \beta < \mathfrak{c}^+\}$. For each $(\alpha, \beta) \in [\mathfrak{c}^+]^2$, since E_α is open in X and $\bigcup r(\mathcal{U}_\beta)$ is dense in X, there is some $k(\alpha, \beta) \in \omega$ such that $E_\alpha \cap V_{\beta,k(\alpha,\beta)} \neq \emptyset$. Applying Erdös–Rado's theorem to the function $f : [\mathfrak{c}^+]^2 \to \omega$ defined by $f((\alpha, \beta)) = k(\alpha, \beta)$, we can take a subset $H \subset \mathfrak{c}^+$ and a $k \in \omega$ such that $|H| = \omega_1$ and $\alpha, \beta \in H$, $\alpha < \beta$ imply $k(\alpha, \beta) = k$ (hence, $E_\alpha \cap V_{\beta,k} \neq \emptyset$). Taking a suitable subset of the well-ordered set H, we may assume that H is order isomorphic to ω_1. Since $\{D_\alpha : \alpha \in H\}$ is not linked and $\overline{E}_\alpha$ is contained in D_α, the family $\mathcal{U} = \{X \setminus \overline{E}_\alpha : \alpha \in H\}$ is an open cover of X. Let $r(\mathcal{U}) = \{W_n : n \in \omega\}$. For each $n \in \omega$, using $r(\mathcal{U}) \prec \mathcal{U}$, we can assign $\alpha_n \in H$ with $W_n \subset X \setminus \overline{E}_{\alpha_n}$ (so, $E_{\alpha_n} \cap W_n = \emptyset$). Let $H' = \{\alpha \in H : \text{for all } n \in \omega, \alpha_n < \alpha\}$. Since $\{D_\alpha : \alpha \in H'\}$ is not linked, there are $\gamma, \delta \in H'$ with $D_\gamma \cap D_\delta = \emptyset$. Fix this $\gamma \in H' \subset H$. Then, $\alpha_n, \gamma \in H$ and $\alpha_n < \gamma$ imply $E_{\alpha_n} \cap V_{\gamma,k} \neq \emptyset$ for all $n \in \omega$. On the other hand, since $\mathcal{U}_\gamma \prec \mathcal{U}$ (because of