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"Counting" spaces and related topics

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ABSTRACT

This is a short survey about the three well-known theories of characteristic classes of singular varieties and the so-called motivic Hirzebruch class, which in a sense "unifies" these three theories as an affirmative answer to a problem posed by Robert MacPherson in the early 1970s.

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1. Introduction

This article is an extended version of the author's talk (with the same title) at "International Conference on Set-Theoretic Topology and Its Application 2015" held at Kanagawa University, August 24–26, 2015. First we start with the basic properties (which even elementary school children know) of counting a finite set, which gives an explanation of why the Euler(–Poincaré) characteristic takes the alternating sum of vertices, edges, faces, and so on. Then we give a survey on the well-known characteristic classes of singular varieties, which "count" spaces in the sense of the above basic properties with a very slight modification. We also give some remarks on motivic Hirzebruch classes [9,6] (for further recent works, see [13,16,37–44], etc.) and related stuff.

2. Euler characteristic revisited

The Euler characteristic of a polyhedra X is defined to be the alternating sum of the numbers of vertices (V), edges (E), faces (F), and so on: $\chi(X) := V - E + F - \cdots$. The following would be a reasonable explanation for why we take *such an alternating sum*.

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First we recall the basic and fundamental properties of the counting of a finite set X, i.e., the cardinality

$$c(X) := |X| = \text{ the number of the elements in the set } X.$$

Certainly the counting c for finite sets satisfies the following basic properties:

(1) $A \cong A'$ (bijection or equipotent) $\Longrightarrow c(A) = c(A')$,
(2) $c(A) = c(A \setminus B) + c(B)$ for $B \subset A$, (this is called *"scissor formula"* or *"motivic"*)
(3) $c(A \times B) = c(A) \cdot c(B)$,
(4) $c(pt) = 1$. (Here pt denotes one point.)

Remark 2.1. Using (1) and (3), one can see that $c(pt) = 0$ or $c(pt) = 1$. If $c(pt) = 0$, $c(A) = c(A \times pt) = c(A) \cdot c(pt) = c(A) \cdot 0 = 0$ for any finite set A. Thus the counting c is a trivial one. Therefore the property (4) $c(pt) = 1$ means that c is non-trivial. Hence the daily-life usual counting c can be said to be a *non-trivial* (4), *multiplicative* (3), *additive* (2) and *set-theoretic invariant* (1), emphasizing the above four properties.

If we consider the following "topological counting" c on the category of certain "nice" topological spaces such that $c(X) \in \mathbb{Z}$ and it satisfies the following four properties:

- $X \cong X'$ (homeomorphism $= \mathcal{TOP} -$ isomorphism) $\Longrightarrow c(X) = c(X')$,
- $c(X) = c(X \setminus Y) + c(Y)$ for $Y \subset X$,
- $c(X \times Y) = c(X) \cdot c(Y)$,
- $c(pt) = 1$,

then one can show that if such a c exists, then we must have that $c(\mathbb{R}) = -1$. Indeed, this can be seen as follows:

$$\mathbb{R} = (-\infty, \infty) = (-\infty, 0) \cup \{0\} \cup (0, \infty) \quad \Longrightarrow \quad c(\mathbb{R}) = c\left((-\infty, 0)\right) + c(\{0\}) + c\left((0, \infty)\right).$$

Since we $(-\infty, 0) \cong \mathbb{R} \cong (0, \infty)$, we obtain

$$c(\mathbb{R}) = c(\mathbb{R}) + c(\{0\}) + c(\mathbb{R}) \quad \Longrightarrow \quad c(\mathbb{R}) = -c(\{0\}) = -1.$$

Hence we have $c(\mathbb{R}^n) = c(\mathbb{R})^n = (-1)^n$.

Theorem 2.2 *(Existence theorem of such a c). The Euler–Poincaré characteristic of the Borel–Moore homology theory $H_*(X)$ or the cohomology with compact support $H^*_c(X)$ gives rise to such a count c, i.e.,*

$$\chi_c(X) := \sum_n (-1)^n \dim H_n(X) = \sum_n (-1)^n \dim H^n_c(X)$$

satisfies the above 4 properties (1), (2), (3) and (4).

Hence, if X is a finite CW-complex with $\sigma_n(X)$ denoting the number of open n-cells, then

$$c(X) = \sum_n (-1)^n \sigma_n(X) = \chi(X)$$

is the Euler–Poincaré characteristic of X. Namely, the topological counting c is uniquely determined and it is the compactly supported Euler–Poincaré characteristic.

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