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Equicontinuity criteria for metric-valued sets of continuous functions ☆

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Combining ideas of Troallic [1] and Cascales, Namioka, and Vera [2], we prove several characterizations of *almost equicontinuity* and *hereditarily almost equicontinuity* for subsets of metric-valued continuous functions when they are defined on a Čech-complete space. We also obtain some applications of these results to topological groups and dynamical systems.

1. Introduction

Let X and (M, d) be a Hausdorff, completely regular space and a metric space, respectively, and let $C(X, M)$ denote the set of all continuous functions from X to M. A subset $G \subseteq C(X, M)$ is said to be *almost equicontinuous* if G is equicontinuous on a dense subset of X. If G is almost equicontinuous for every closed nonempty subset of X, then it is said that G is *hereditarily almost equicontinuous*. The main goal of this paper is to extend to arbitrary topological spaces these two important notions, which were introduced in the setting of topological dynamics studying the enveloping semigroup of a flow [3–5].

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In addition to their intrinsic academic interest, it turns out that these two concepts have found application in other different settings as it will be made clear in the sequel. First, we shall provide some basic notions and terminology.

Given $F \subseteq X$, the symbol $t_p(F)$ (resp. $t_\infty(F)$) will denote the topology, on $C(X, M)$, of pointwise convergence (resp. uniform convergence) on F. For a set G of functions from X to M and $Z \subseteq X$, the symbol $G|_Z$ will denote the set $\{g|_Z : g \in G\}$. We denote by $\overline{G}^{M^X}$ the closure of G in the Tychonoff product space M^X.

The symbolism $(F, t_p(\overline{G}^{M^X}))$ will denote the set F equipped with the weak topology generated by the functions in $\overline{G}^{M^X}|_F$. In like manner, the symbol $[A]^{\leq\omega}$ will denote the set of all countable subsets of A. A topological space X is said to be *Čech-complete* if it is a G_δ-subset of its Stone–Čech compactification. The family of Čech-complete spaces includes all complete metric spaces and all locally compact spaces. Several quotient spaces are used along the paper. For the reader's sake, a detailed description of them is presented at the Appendix.

We now formulate our main results.

Theorem A. *Let X and (M, d) be a Čech-complete space and a separable metric space, respectively, and let $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Consider the following three properties:*

(a) *G is almost equicontinuous.*
(b) *There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{M^X})|_F$ is metrizable.*
(c) *There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{M^X}))$ is Lindelöf.*

Then $(b) \Rightarrow (c) \Rightarrow (a)$. If X is also a hereditarily Lindelöf space, then all conditions are equivalent.

Next result characterizes hereditarily almost equicontinuous families of functions defined on a Čech-complete space (this question has been studied in detail in [6] for compact spaces).

Theorem B. *Let X and (M, d) be a Čech-complete space and a metric space, respectively, and let $G \subseteq C(X, M)$ such that $\overline{G}^{M^X}$ is compact. Then the following conditions are equivalent:*

(a) *G is hereditarily almost equicontinuous.*
(b) *L is hereditarily almost equicontinuous on F, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.*
(c) *$(\overline{L}^{M^X})|_F$ is metrizable, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.*
(d) *$(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [G]^{\leq\omega}$ and F a separable and compact subset of X.*

Remark 1.3. If G is a subset of $C(X, M)$ such that $K \stackrel{\text{def}}{=} \overline{G}^{M^X}$ is contained in $C(X, M)$, then the implication $(c) \Rightarrow (a)$ in Theorem B provides a different proof of the celebrated Namioka Theorem [7, Theorem 2.3]. Indeed, given any $L \in [G]^{\leq\omega}$ and any separable compact subset F of X, since $K \subseteq C(X, M)$ and F is separable, it follows that $((\overline{L}^{M^X})|_F, t_p(F))$ is metrizable. Thus G (and therefore K) is hereditarily almost equicontinuous.

Corollary 1.4. *With the same hypothesis of Theorem B, consider the following three properties:*

(a) *G is hereditarily almost equicontinuous.*
(b) *G is hereditarily almost equicontinuous on F, for all F a separable and compact subset of X.*
(c) *$(F, t_p(\overline{G}^{M^X}))$ is Lindelöf, for all F a separable and compact subset of X.*

Then $(a) \Leftrightarrow (b) \Leftarrow (c)$.