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Dynamics of typical Baire-1 functions

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ABSTRACT

Let M be the Cantor space or an n-manifold with $B_1(M, M)$ the set of Baire-1 self-maps of M. We prove the following:

1. For the typical $f \in B_1(M, M)$, the maps $x \longmapsto \omega(x, f)$ and $x \longmapsto \tau(x, f)$ taking x to its ω-limit set and trajectory, respectively, are continuous at a typical point $x \in M$.
2. If $s > 0$, then for the typical $(x, f) \in M \times B_1(M, M)$, the Hausdorff s-dimensional measure of $\omega(x, f)$ is zero.
3. If G_1 is a residual subset of M, then there is a residual set of points $G_2 \subset M \times B_1(M, M)$ all of which generate as their ω-limit set a particular, unique type of adding machine, and if $(x, f) \in G_2$, then $\omega(x, f) \subset G_1$.

1. Introduction

A considerable amount of recent research has been dedicated to studying the iterative behavior of continuous functions mapping a compact space into itself. Here, we study the iterative behavior of Baire-1 functions B_1 mapping M, an n-manifold or the Cantor space, into itself.

Let $B_1(M, M)$ be the set of Baire-1 functions from M to M. Recall that a function $f : X \to Y$ is in the first class of Baire if it is the pointwise limit of a sequence of continuous functions from X to Y. Baire-1 functions are ubiquitous in analysis. The class of derivatives, approximately continuous functions, semi-continuous functions and functions of bounded variation are all examples of Baire-1 functions [6]. Let K be the collection of non-empty closed sets in M endowed with the Hausdorff metric. Here, we investigate the behavior of $\omega : M \times B_1(M, M) \to K$ at the typical point (x, f) found in the complete metric space $M \times B_1(M, M)$, as well as the measure of the ω-limit set $\omega(x, f)$ and the stability of the trajectory $\tau(x, f) = \{x, f(x), f(f(x)), ...\}$. These concerns, and more surprisingly, our results, are similar to some of those found in [1,11,13], where the

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authors concern themselves with the behavior of typical continuous self-maps of compact spaces. We will see that for the typical $f \in B_1(M, M)$, the maps $x \longmapsto \omega(x, f)$ and $x \longmapsto \tau(x, f)$ taking x to its ω-limit set and trajectory, respectively, are continuous at a typical point $x \in M$. Speaking loosely, the separation of trajectories generally associated with chaotic behavior is exceptional behavior in $M \times B_1(M, M)$. By using a generalization of Hausdorff measure [8], we will prove a bit more than the assertion that if $s > 0$, then for the typical $(x, f) \in M \times B_1(M, M)$, the Hausdorff s-dimensional measure of $\omega(x, f)$ is zero. We conclude by showing that not only is $\{(x, f) \in M \times B_1(M, M) : \omega(x, f) \text{ is periodic}\}$ dense in $M \times B_1(M, M)$, but that $\{(x, f) \in M \times B_1(M, M) : \omega(x, f) \text{ is an } \infty\text{-adic adding machine}\}$ is residual in $M \times B_1(M, M)$. This result is very similar to one found in [9], where the authors show that if f is typical in $C(M, M)$, then, for the typical x in M, f restricted to $\omega(x, f)$ is an odometer of type ∞. This particular result is considerably strengthened in [2], in the case that M is the Cantor space. Specifically, Bernardes and Darji show that for the typical $f \in C(\{0, 1\}^{\mathbb{N}})$, the restriction of f to every ω-limit set $\omega(x, f)$ is an odometer of type ∞.

Adding machines are a particular class of minimal sets also referred to as solenoids or odometers. These are all Cantor sets, where every point is not only strongly recurrent, but also regularly recurrent [3,4]. Adding machines are a fundamental component of one-dimensional systems. They are a basic component of Blokh's spectral decomposition theorem [5], and critical to Nitecki's analysis of the dynamics of piecewise monotone maps [12]. A recent paper of Block and Keesling [4] gives topological characterizations of adding machines. Among them is one that characterizes when two adding machines are topologically conjugate. This result, which is found in section 2, is essential to our analysis.

2. Preliminaries

Unless otherwise stated, M will be either an n-manifold with or without boundary, or the Cantor space. All of the manifolds will be compact, and the main property that we use is that each point of a manifold has an arbitrarily small neighborhood whose closure is homeomorphic to I^n, the unit cube in $\mathbb{R}^n$ [10]. Should M be the Cantor space, then every point has an arbitrarily small clopen neighborhood which is homeomorphic to the Cantor space. Call a subset U of M an n-cube if U is homeomorphic to I^n, and U has nonempty interior relative to M. If M is the Cantor space, then a 0-cube is a nonempty clopen subset of M. Let $(S)^0$, $\overline{S}$, δS and $diam(S)$ represent the interior, closure, boundary and diameter, respectively, of a set S contained in M. Extensive use is made of the following version of Tietze's extension theorem when perturbing Baire-1 functions on n-cubes:

Theorem 1. *([10]) If U is the Cantor space or an n-cube and f is a continuous function from a closed set C of U into U, then f can be extended to a continuous function whose domain is all of U, and $f(U) \subseteq U$.*

We make repeated use of the following properties of Baire-1 functions:

Theorem 2. *([6]) Let $f : M \to M$ be a Baire-1 function, and let $g : M \to M$ be continuous. Then $g \circ f$ and $f \circ g$ are both contained in the first class of Baire. Moreover, there exists G a residual subset of M such that f is continuous at every element x in G. Finally, if $\{h_n\} \subset B_1(M, M)$, and $h_n \to h$ [unif], then $h \in B_1(M, M)$, too.*

We shall be concerned with the class $B_1(M, M)$, and the iterative properties that this class of functions possesses. For f in $B_1(M, M)$ and any integer $n \geq 1$, f^n denotes the n-th iterate of f. For each x in M, we call $\tau(x, f) = \{f^n(x)\}_{n=0}^{\infty}$ the trajectory of x, and take the set of all subsequential limits of the trajectory $\tau(x, f)$ to be the ω-limit set of f generated by x, and write $\omega(x, f)$. Equivalently, $\omega(x, f) = \cap_{m \geq 0} \overline{\cup_{k \geq m} f^k(x)}$.

Along with M, we work in four metric spaces. In each of these spaces, $B_\varepsilon(\circ)$ denotes an open ball of radius ε centered at $\circ$; the nature of $\circ$ will indicate which space we are considering. Since $B_1(M, M)$ is