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Spherical classes in some finite loop spaces of spheres

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A B S T R A C T

We completely determine spherical classes in single, double and triple loop spaces of spheres. We also show that for $l \geqslant 4$ and $n > 1$, there exists no spherical class in $H_*\Omega^l S^{l+n}$ in dimensions more than $2^l n + 2^{l-1}(l-2) + 2$. This bound may not be optimal, but reduces the computation of spherical classes in $H_*\Omega^l S^{l+n}$ to verifying finite number of cases. We also provide a table for possible spherical classes in $H_*\Omega^l S^{l+n}$ with $3 < l < 9$. Our results, provide positive evidence for the truth of Eccles conjecture.

1. Introduction

For a pointed space X, let $QX = \operatorname{colim} \Omega^i\Sigma^i X$ be the infinite loop space associated to $\Sigma^\infty X$. Curtis conjecture on spherical classes in H_*QS^0 reads as following.

Conjecture 1.1. *([5, Theorem 7.1]) For $n > 0$, only Hopf invariant one and Kervaire invariant one elements map nontrivially under the unstable Hurewicz homomorphism*

$$ {}_2\pi_n^s \simeq {}_2\pi_n QS^0 \to H_*QS^0. $$

Throughout the paper, we shall work at the prime 2, the homology will be $\mathbb{Z}/2$-homology; we write ${}_2\pi_*^s$ and ${}_2\pi_*$ for the 2-component of π_*^s and π_* respectively. We also write H_* for $H_*(-;\mathbb{Z}/2)$.

A more generalised question, is the determination of the image of the Hurewicz homomorphism $\pi_*\Omega^\infty E \to H_*\Omega^\infty E$ where E is a suitable spectrum. The aim of this paper is to continue this investigation when $E = \Sigma^\infty S^n$ with $n > 0$ by means of examining the conjecture on finite loops spaces associated to spheres. Let us recall a variant of Curtis conjecture, due to Eccles, which may be stated as follows.

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Conjecture 1.2 *(Eccles conjecture). Let X be a path connected CW-complex with finitely generated homology. For $n > 0$, suppose $h(f) \neq 0$ where ${}_2\pi_n^s X \simeq {}_2\pi_n QX \to H_*QX$ is the unstable Hurewicz homomorphism. Then, the stable adjoint of f either is detected by homology or is detected by a primary operation in its mapping cone.*

Note that the stable adjoint of f being detected by homology means that $h(f) \in H_*QX$ is stably spherical, i.e. it survives under homology suspension $H_*QX \to H_*X$ induced by $\Sigma^\infty QX \to \Sigma^\infty X$ given by the adjoint of the identity $QX \to QX$.

The above conjectures make predictions about the image of spherical classes under the unstable Hurewicz homomorphism $h : {}_2\pi_*^s X \simeq {}_2\pi_* QX \to H_*QX$. By Freudenthal suspension theorem, for any $f \in \pi_n QX$, depending on the connectivity of X, we may find some nonnegative integer i so that f does pull back to $\pi_n\Omega^i\Sigma^i X$. So, it is natural to ask about the image of the Hurewicz homomorphism $\pi_n\Omega^i\Sigma^i X \to H_n\Omega^i\Sigma^i X$. Note that for each $i > 0$ there exists an obvious commutative diagram

$$\begin{tikzcd}
\pi_n\Omega^i\Sigma^i X \arrow[r, "h"] \arrow[d] & H_n\Omega^i\Sigma^i X \arrow[d] \\
\pi_n QX \arrow[r, "h"] & H_n QX.
\end{tikzcd}$$

It is then natural to look for spherical classes in $H_*\Omega^i\Sigma^i X$ for all i in order to give some answer to the above conjectures.

2. Statement of results

Let's recall a related result which was mainly proved by means of geometric tools.

Theorem 2.1. *Consider the following lattice where (a, c) corresponds to $H_*\Omega^a S^c$*

$$\begin{tikzcd}
(1,15) \arrow[d, dotted, no head] \arrow[ddrrrrr, dotted] & & & & & \\
(1,9) \arrow[d, dotted, no head] & & & & & \\
(1,8) \arrow[r, dotted] \arrow[d, dotted, no head] \arrow[dr, dotted] & (2,9) \arrow[rrrr, dotted] & & & & (8,15) \\
(1,7) \arrow[d, dotted, no head] \arrow[drrr, dotted] & (2,8) \arrow[drrr, dotted] & & & & \\
(1,4) \arrow[r, dotted] \arrow[d, dotted, no head] \arrow[ddrr, dotted] & (2,5) \arrow[rr, dotted] & & (4,7) \arrow[r, dotted] & (5,8) & \\
(1,3) \arrow[d, dotted, no head] \arrow[dr, dotted] & & & & & \\
(1,2) \arrow[r, dotted] \arrow[d, dotted, no head] & (2,3) \arrow[r, dotted] & (3,4) & & & \\
(1,1) \arrow[r, dotted] & (2,2) & & & &
\end{tikzcd}$$