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Open quotients of trivial vector bundles ☆

Pedro Resende *, João Paulo Santos

Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

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Given an arbitrary topological complex vector space A, a *quotient vector bundle* for A is a quotient of a trivial vector bundle $\pi_2 : A \times X \to X$ by a fiberwise linear continuous open surjection. We show that this notion subsumes that of a Banach bundle over a locally compact Hausdorff space X. Hyperspaces consisting of linear subspaces of A, topologized with natural topologies that include the lower Vietoris topology and the Fell topology, provide classifying spaces for various classes of quotient vector bundles, in a way that generalizes the classification of locally trivial vector bundles by Grassmannians. If A is normed, a finer hyperspace topology is introduced that classifies bundles with continuous norm, including Banach bundles, and such that bundles of constant finite rank must be locally trivial.

1. Introduction

The reduced C*-algebra $C^*_{\mathrm{red}}(G)$ of a locally compact Hausdorff étale groupoid G (see [11,10]), whose elements can be regarded as complex valued functions on G, can be "twisted" by considering instead the reduced C*-algebra $C^*_{\mathrm{red}}(\pi)$ of a Fell line bundle $\pi : E \to G$, where now the elements of the algebra are identified with sections of the bundle; see [12,5]. This provides one way of generalizing to C*-algebras the notion of Cartan subalgebra of a Von Neumann algebra [1]: *Cartan pairs* (A, B), consisting of a C*-algebra A and a suitable abelian subalgebra B, correspond bijectively to a certain class of étale groupoids with Fell line bundles on them [12].

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* Corresponding author.
E-mail addresses: pmr@math.tecnico.ulisboa.pt (P. Resende), jsantos@math.tecnico.ulisboa.pt (J.P. Santos).

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The motivation for the present paper stemmed from studying the construction of a Fell bundle from a Cartan pair, in particular in an attempt to generalize the class of groupoids to which it applies by taking into account that both a C*-algebra A and an étale groupoid G have associated quantales $\operatorname{Max} A$ [4,7,8] and $\Omega(G)$ [13], respectively, the former consisting of all the closed linear subspaces of A and the latter being the topology of G. In doing so it became evident that it is useful to study bundles whose construction is based on a preexisting object of global sections, such as the C*-algebra A of a Cartan pair, in a way that in fact is independent of the algebraic structure of groupoids, but which instead applies to Banach bundles to begin with, and indeed to more general bundles. So the original endeavor has naturally been split into several parts, of which the present paper is the first one, where no further mention of C*-algebras or groupoids will be made. Instead the bulk of this paper will deal with completely general topological vector spaces, and on occasion locally convex or normed spaces.

If A is a topological vector space and X is any topological space, then $\pi_2 : A \times X \to X$ is a trivial vector bundle on X. The vector bundles studied in this paper, termed *quotient vector bundles*, consist of those bundles $\pi : E \to X$ that arise as quotients of trivial bundles by a fiberwise linear continuous open surjection q:

$$\begin{tikzcd} A \times X \arrow[rr, two heads, "q"] \arrow[dr, "\pi_2"'] & & E \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

We shall see that any Banach bundle $\pi : E \to X$ (in particular, any finite rank locally trivial vector bundle) on a locally compact Hausdorff space is of this kind, where A can be taken to be the space $C_0(\pi)$ of continuous sections vanishing at infinity.

Although now without any quantale structure (since A is not even an algebra), the set $\operatorname{Max} A$ of all the closed linear subspaces of A plays an important role: equipped with suitable topologies it provides a notion of classifying space for quotient vector bundles. Concretely, these are obtained by pullback along continuous maps

$$\kappa : X \to \operatorname{Max} A$$

(or, even more generally, maps into $\operatorname{Sub} A$, the space of all the linear subspaces) of a universal bundle $\pi_A : E_A \to \operatorname{Max} A$:

$$\begin{tikzcd} E_A \times_{\operatorname{Max} A} X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & E_A \arrow[d, "\pi_A"] \\ X \arrow[r, "\kappa"'] & \operatorname{Max} A \end{tikzcd}$$

In this paper we study three topologies on $\operatorname{Max} A$. Perhaps surprisingly, two of them are well known hyperspace topologies:

- The (relative) lower Vietoris topology [9,14] classifies all the quotient vector bundles with Hausdorff fibers (the restriction on the fibers disappears if we use $\operatorname{Sub} A$ instead of $\operatorname{Max} A$).
- The topology of Fell [2] classifies the quotient vector bundles whose zero section is closed — at least provided both A and X are first countable.

If A is normed its quotient vector bundles $\pi : E \to X$ are naturally equipped with an upper semicontinuous norm $\| \ \| : E \to \mathbb{R}$. In this case a third topology on $\operatorname{Max} A$, referred to as the *closed balls topology*, coarser