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The geometric intersection number of simple closed curves on a surface and symplectic expansions of free groups

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ABSTRACT

For two oriented simple closed curves on a compact orientable surface with a connected boundary we introduce a simple computation of a value in the first homology group of the surface, which detects in some cases that the geometric intersection number of the curves is greater than zero when their algebraic intersection number is zero. The value, computed from two elements of the fundamental group of the surface corresponding to the curves, is found in the difference between one of the elements and its image of the action of Dehn twist along the other. To give a description of the difference symplectic expansions of free groups is an effective tool, since we have an explicit formula for the action of Dehn twist on the target space of the expansion due to N. Kawazumi and Y. Kuno.

1. Introduction

Let $\Sigma = \Sigma_{g,1}$ be a compact oriented surface of genus $g \geq 1$ with a connected boundary. Let α and β be oriented simple closed curves on Σ. We always assume that the intersections of the curves are transverse double points. The geometric intersection number of α with β, which we denote by $i_G(\alpha, \beta)$, is the minimal number of intersection points of α with any simple closed curve on Σ freely homotopic to β. The algebraic intersection number of α with β, which we denote by $i_A(\alpha, \beta)$, is the sum of signs of intersection points of α with β, where the sign of an intersection point of α with β is $+1$ when a pair of tangent vectors of α and β in this order is consistent with an oriented basis for Σ, otherwise the sign is -1.

In this paper we describe an algebraic value associated with two simple closed curves α and β on Σ, which may detect that $i_G(\alpha, \beta)$ is greater than zero when $i_A(\alpha, \beta)$ is zero. Our strategy is as follows: The geometric intersection number of two simple closed curves on a surface is closely related to the change of the homotopy type of one of the curves by performing the Dehn twist along the other. In particular it is known (e.g. [1]) that $i_G(\alpha, \beta) = 0$ if and only if the Dehn twist along one of the curves does not change the homotopy type of the other. In the case that one of the curves, say α, is non-separating, this can be rephrased that there

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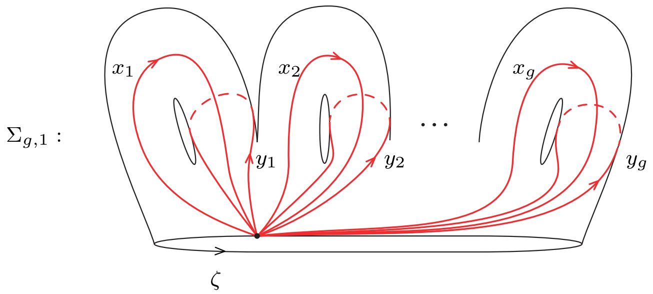

Fig. 1. Symplectic generators of π.

exists a based homotopy class b of a based loop freely homotopic to β such that $t_\alpha(b) = b$. To analyze the action of Dehn twist on the fundamental group of a surface, the symplectic expansion, a certain type of (generalized) Magnus expansion defined by G. Massuyeau [5], of free groups is an effective tool, since we have an explicit formula describing the action of Dehn twist on the target space of the expansion due to N. Kawazumi and Y. Kuno. (See §2 for details.) Applying the formula we can compute differences between expansion of $t_\alpha(b)$ and of b. Since the first degree part of the expansions of $t_\alpha(b)$ and b coincide because of the condition $i_A(\alpha, \beta) = 0$, we will focus on the second degree part of them.

In the rest of this section we mention our main theorem and demonstrate an application of the theorem. In §2, we recall the terminology of Kawazumi–Kuno's theory developed in [3], and prepare some propositions for use in §3, in which we give a proof of the main theorem.

1.1. Main result

The fundamental group $\pi = \pi_1(\Sigma, p)$ of Σ with a point p on $\partial\Sigma$ is a free group of rank $2g$ and the first homology group of Σ with coefficients in $\mathbb{Q}$, $H = H_1(\Sigma, \mathbb{Q}) = \pi/[\pi, \pi] \otimes \mathbb{Q}$, is a free abelian group of rank $2g$. For $x \in \pi$, we denote by $|x|$ the element of H corresponding to x via the abelianization of π.

Let $\{x_1, y_1, \ldots, x_g, y_g\}$ be symplectic generators of π as in Fig. 1. Putting $X_i = |x_i|$, $Y_i = |y_i|$ $(i = 1, 2, \ldots, g)$, we have a symplectic basis $\{X_1, Y_1, \ldots, X_g, Y_g\}$ of H, namely the basis satisfies $X_i \cdot Y_j = \delta_{ij}$, $X_i \cdot X_j = Y_i \cdot Y_j = 0$ for $1 \le i, j \le g$, where $\cdot : H \times H \to \mathbb{Q}$ is the intersection form on H.

With respect to the symplectic generators of π, we define a map $\ell : \pi \to H \wedge H$ by the following three rules:

i) $\ell(1) = 0$,
ii) $\ell(x_i) = \frac{1}{2} X_i \wedge Y_i$, $\ell(y_i) = -\frac{1}{2} X_i \wedge Y_i$ $(i = 1, 2, \ldots, g)$,
iii) For all $g, h \in \pi$, $\ell(gh) = \ell(g) + \ell(h) + \frac{1}{2}|g| \wedge |h|$.

Note that we obtain $\ell(g^{-1}) = -\ell(g)$ for all $g \in \pi$ from i) and iii).

We consider that $H \wedge H$ acts on H as follows: For $X \wedge Y \in H \wedge H$ and $Z \in H$,

$$(X \wedge Y)(Z) := (Z \cdot X)Y - (Z \cdot Y)X.$$

Theorem 1.1. *Let α and β be oriented simple closed curves on Σ with $i_A(\alpha, \beta) = 0$. If $\ell(a)(|b|) + \ell(b)(|a|)$ is not an element in $\mathbb{Z}|a| + \mathbb{Z}|b| \subset H$, where a and b are based homotopy classes of based loops freely homotopic to α and β respectively, then $i_G(\alpha, \beta) > 0$.*

Remark 1.1. (1) When both of the curves are separating on Σ, the theorem is trivial since $\ell(a)(|b|) + \ell(b)(|a|) = 0 \in \mathbb{Z}|a| + \mathbb{Z}|b|$.