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Topological gyrogroups: Generalization of topological groups

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ABSTRACT

Left Bol loops with the A_ℓ-property or gyrogroups are generalization of groups which do not explicitly have associativity. In this work, we define topological gyrogroups and study some properties of them. In spite of having a weaker algebraic form, topological gyrogroups carry almost the same basic properties owned by topological groups. In particular, we prove that being T_0 and T_3 are equivalent in topological gyrogroups. Furthermore, a topological gyrogroup is first countable if and only if it is premetrizable. Finally, a direct product of topological gyrogroups is a topological gyrogroup.

1. Introduction

A loop is an algebraic structure first introduced by R. H. Bruck (see [4]). It is a set G equipped with a binary operation $\cdot : G \times G \to G$ making inverse operation possible and identity exists. On the other hand, a gyrogroup is also a relaxation of a group which the associativity condition has been replaced by a weaker one. This concept is originated from the study of c-ball of relativistically admissible velocities with Einstein velocity addition as mentioned by A. A. Ungar in [18].

There is an equivalence between a type of loops called left Bol loops with the A_ℓ-property and gyrogroups. This phenomenon has been first observed in [10]. However, there are many people working on these two subjects using different names. To simplify our work, we will adopt the notion of gyrogroups, and formulate everything in their terminology.

Next, a topological group is a mathematical object studied for a long time since the last century. This is a group endowed with a topology permitting the continuity of its binary and inverse operations. There are many interesting results coming out during its long history. As a consequence, the notion of topological groups are widely generalized into topological monoids, topological semigroups, topological loops, topological quasigroups and so on. Although there are some results on topological quasigroups and topological loops (see [1,5,6,9,11]), there are still not very much research comparing with other similar areas. So, it is worth

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extending the idea of topological groups to topological gyrogroups as gyrogroups with a topology such that its binary operation is jointly continuous and the operation of taking the inverse is continuous.

2. Preliminaries

Throughout this paper, we will assume that the reader is familiar with basic results in abstract algebra and point-set topology. For less common terms, we will provide definitions and necessary facts.

Let G be a nonempty set, and let $\oplus : G \times G \to G$ be a binary operation on G. Then the pair $(G, \oplus)$ is called a *groupoid* or a *magma*. A function f from a groupoid $(G_1, \oplus_1)$ to a groupoid $(G_2, \oplus_2)$ is said to be a *groupoid homomorphism* if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any elements $x, y \in G_1$. In addition, a bijective groupoid homomorphism from a groupoid $(G, \oplus)$ to itself will be called a *groupoid automorphism*. We will write $Aut(G, \oplus)$ for the set of all automorphisms of a groupoid $(G, \oplus)$.

By following the language of [15] and [18], we define a *gyrogroup* to be a groupoid $(G, \oplus)$ which has the following properties:

1. There exists a unique *identity* element $0 \in G$ such that

$$0 \oplus x = x = x \oplus 0 \qquad \text{for all } x \in G$$

2. For each $x \in G$, there exists a unique *inverse* element $\ominus x \in G$ such that

$$\ominus x \oplus x = 0 = x \oplus (\ominus x)$$

3. For any $x, y \in G$, there exists $\text{gyr}[x, y] \in Aut(G, \oplus)$ with the property that

$$x \oplus (y \oplus z) = (x \oplus y) \oplus \text{gyr}[x, y](z) \qquad \text{for all } z \in G$$

4. For any $x, y \in G$, we obtain that $\text{gyr}[x \oplus y, y] = \text{gyr}[x, y]$.

A gyrogroup which is not a group does exist in general. We give such examples in the next section. Moreover, [15] defined a *subgyrogroup* of a gyrogroup $(G, \oplus)$ to be a nonempty subset A of G such that the following hold:

1. The restriction $\oplus_{|A \times A}$ is a binary operation on A, i.e. $(A, \oplus_{|A \times A})$ is a groupoid
2. For any $x, y \in A$, the restriction of $\text{gyr}[x, y]$ to A, $\text{gyr}[x, y]_{|A} : A \to \text{gyr}[x, y](A)$, is a bijective homomorphism
3. $(A, \oplus_{|A \times A})$ is a gyrogroup.

Proposition 1 *(Proposition 4.1 of [15]). Let $(G, \oplus)$ be a gyrogroup, and let A be a nonempty subset of G. Then A is a subgyrogroup if and only if the following are true:*

1. *For any $x \in A$, $\ominus x \in A$.*
2. *For any $x, y \in A$, $x \oplus y \in A$.*

To compactify the cancellation law in gyrogroups, we first define the *coaddition* to be the binary operation $\boxplus : G \times G \to G$ such that $x \boxplus y = x \oplus \text{gyr}[x, \ominus y](y)$ for any $x, y \in G$.

Theorem 1 *(Theorem 2.10 and Theorem 2.22 of [18]). Let $(G, \oplus)$ be a gyrogroup. Then for any $x, y, z \in G$, we obtain the following:*