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Sphere eversion from the viewpoint of generic homotopy [☆]

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ABSTRACT

In 1958, Smale proved that any immersions of S^2 to $\mathbb{R}^3$ are regularly homotopic. This means that we can turn an embedded sphere in $\mathbb{R}^3$ inside out by a regular homotopy. After Smale showed his result without visualization, many people visualized sphere eversions, in various ways. In this paper, we construct a sphere eversion by lifting a "simple" generic homotopy of S^2 to $\mathbb{R}^2$ to a generic regular homotopy of S^2 to $\mathbb{R}^3$. By doing so, our eversion is simple in terms of deformation of the contour generators of immersed spheres. We also visualize the 3-dimensional interlinking of the contour generators and the self-intersections of each stage of immersed spheres.

1. Introduction

Smale [21], James and Thomas [12] completely classified the immersions of closed surfaces in $\mathbb{R}^3$ up to regular homotopy. In particular, Smale proved that any immersions of S^2 to $\mathbb{R}^3$ are regularly homotopic. This means that one can evert an embedded S^2 in $\mathbb{R}^3$, that is, turn a sphere inside out by allowing self-intersections but no holes, rips, creases or pinches. After Smale's proof by using h-principle, without actual construction, many people visualized sphere eversions in various ways. See [1,2,4,6–8,13–17,19] for example.

In this paper, we give another version of a sphere eversion (see Figs. 5, 11–13). We construct our sphere eversion by lifting a generic homotopy of S^2 to $\mathbb{R}^2$ to a generic regular homotopy of S^2 to $\mathbb{R}^3$. Obviously, seeing a sequence of (stable) maps in $\mathbb{R}^2$ is easier than seeing those of (generic) immersions in $\mathbb{R}^3$. Our drawing has two advantages over those already done: (i) simpleness of the sequence of the contours and contour generators of immersed spheres during the eversion, and (ii) complete visualization of the self-intersections of the immersions, without shading and opening windows.

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First advantage of our eversion is the simpleness of the sequence of contours (Fig. 5) and contour generators (Fig. 14): Let S be an immersed surface in $\mathbb{R}^3$ and consider the projection of S to the XY-plane. The *contour generator* of S with respect to this projection is those points in S that has a tangential line parallel to the Z-axes. The projection image of contour generator is called the *contour*. (For a precise definition, see Section 2.) Note that even if S is embedded, the contour may have cusps and self-intersections. See Figs. 1 and 4.

To convince ourselves that Fig. 5 gives a blueprint of sphere eversion, it is necessary to check that a generic homotopy obtained from Fig. 5 lifts to a regular homotopy. From Blank and Curley's theorem ([3], Theorem 3.1), we obtain a necessary and sufficient condition of the existence of regular homotopy lift of a given generic homotopy (see Theorem 3.2). By using our condition, we see that a sequence of contour generators (i.e. lifts of contours in Fig. 5) gives a frame of a sequence of immersed spheres corresponding to a sphere eversion.

Using Fig. 5 as a blueprint, we construct the whole sphere eversion as a generic regular homotopy. First we make a dent and push down the neighborhood of the north pole (Fig. 11, (i)–(iii)). Then Fig. 11 (iii) to (iv) is the essential part (from Figs. 12 through 13). Finally, it is easy to see Fig. 11 (iv)–(vi) completing the eversion. The essential part happens in the annular neighborhood R of a latitude. Contain the annulus R in a cube so that the boundaries are squares. Then, we describe the essential part of our eversion fixing the boundary of R from Figs. 12 through 13. In Figs. 12 and 13, we should paint the contour generator in red and blue according to which side of the sphere is showing and we draw the self-intersections in thick lines (see the web version for color). Through the process, we slice the surfaces by planes parpendicular to the paper, and open the cuts so that both sections are visible to the reader. In particular, we can draw the sphere eversion without shading the surface or opening windows, which is the second advantage of our version of sphere eversion. Finally, in Fig. 14, we superimpose the sequence of the contour generators (they should be painted red and blue) and the self-intersection curves (they should be painted green).

This paper is organized as follows. In Section 2, as preliminaries, we give the definitions and see properties of a stable map, a generic homotopy, an immersion and a generic regular homotopy. In Section 3, to construct a sphere eversion from a generic homotopy, we give a necessary and sufficient condition of the existence of regular homotopy lift of a given generic homotopy. In Section 4, we construct and describe our sphere eversion $\tilde{F} : S^2 \times [0,1] \to \mathbb{R}^3$ as a generic regular homotopy lift of our "simple" generic homotopy.

Throughout this paper, all manifolds and maps are differentiable of class C^∞.

2. Preliminaries

2.1. Stable map and generic homotopy

In this subsection, we give the definition and see properties of a stable map and a generic homotopy.

Let $f : M \to N$ be a map of a closed connected manifold M to a connected manifold N. We denote the set of such maps by $C^\infty(M, N)$, which is equipped with the Whitney C^∞-topology. A map f is said to be a *stable map* if there exists, in $C^\infty(M, N)$, an open neighborhood U of f such that for any $g \in U$, g is right-left equivalent to f, i.e. there exist two diffeomorphisms $\Phi : M \to M$ and $\varphi : N \to N$ such that the diagram is commutative.

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & M \\ f\downarrow & & \downarrow g \\ N & \xrightarrow[\varphi]{} & N \end{array}$$