Topology and its Applications 223 (2017) 30–49

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Free topological vector spaces

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ARTICLE INFO

Article history:
Received 13 May 2016
Received in revised form 26 February 2017
Accepted 4 March 2017
Available online 9 March 2017

MSC:
46A03
54A25
54D50

Keywords:
Free topological vector space
Free locally convex space
k_ω-Space
Barreled space

ABSTRACT

In this paper the free topological vector space $\mathbb{V}(X)$ over a Tychonoff space X is defined and studied. It is proved that $\mathbb{V}(X)$ is a k_ω-space if and only if X is a k_ω-space. If X is infinite, then $\mathbb{V}(X)$ contains a closed vector subspace which is topologically isomorphic to $\mathbb{V}(\mathbb{N})$. It is proved that for X a k-space, the free topological vector space $\mathbb{V}(X)$ is locally convex if and only if X is discrete and countable. The free topological vector space $\mathbb{V}(X)$ is shown to be metrizable if and only if X is finite if and only if $\mathbb{V}(X)$ is locally compact. Further, $\mathbb{V}(X)$ is a cosmic space if and only if X is a cosmic space if and only if the free locally convex space $L(X)$ on X is a cosmic space. If a sequential (for example, metrizable) space Y is such that the free locally convex space $L(Y)$ embeds as a subspace of $\mathbb{V}(X)$, then Y is a discrete space. It is proved that $\mathbb{V}(X)$ is a barreled topological vector space if and only if X is discrete. This result is applied to free locally convex spaces $L(X)$ over a Tychonoff space X by showing that: (1) $L(X)$ is quasibarreled if and only if $L(X)$ is barreled if and only if X is discrete, and (2) $L(X)$ is a Baire space if and only if X is finite.

1. Introduction

Until recently almost all papers in topological vector spaces restricted themselves to locally convex spaces. However in recent years a number of questions about non-locally convex vector spaces have arisen.

All topological spaces are assumed here to be Tychonoff and all vector spaces are over the field of real numbers $\mathbb{R}$. The free topological group $F(X)$, the free abelian topological group $A(X)$ and the free locally convex space $L(X)$ over a Tychonoff space X were introduced by Markov [18] and intensively studied over the last half-century, see for example [1,9,13,16,26,29]. It has been known for half a century that the (Freyd) Adjoint Functor Theorem ([17] or Theorem A3.60 of [10]) implies the existence and uniqueness of $F(X)$,

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http://dx.doi.org/10.1016/j.topol.2017.03.006

$A(X)$ and $L(X)$. This paper focuses on free topological vector spaces. One surprising fact is that free topological vector spaces in some respect behave better than free locally convex spaces.

2. Basic properties of free topological vector spaces

Definition 2.1. The *free topological vector space* $\mathbb{V}(X)$ over a Tychonoff space X is a pair consisting of a topological vector space $\mathbb{V}(X)$ and a continuous map $i = i_X : X \to \mathbb{V}(X)$ such that every continuous map f from X to a topological vector space (tvs) E gives rise to a unique continuous linear operator $\bar{f} : \mathbb{V}(X) \to E$ with $f = \bar{f} \circ i$.

In analogy with the Graev free abelian topological group over a Tychonoff space X with a distinguished point p, we can define the Graev free topological vector space $\mathbb{V}_G(X,p)$ over (X,p).

Definition 2.2. The *Graev free topological vector space* $\mathbb{V}_G(X,e)$ over a Tychonoff space X with a distinguished point e is a pair consisting of a topological vector space $\mathbb{V}_G(X,e)$ and a continuous map $i = i_X : X \to \mathbb{V}_G(X,e)$ such that $i(e) = 0$ and every continuous map f from X to a topological vector space E with $f(e) = 0$ gives rise to a unique continuous linear operator $\bar{f} : \mathbb{V}_G(X,e) \to E$ with $f = \bar{f} \circ i$.

Set $\mathbb{N} := \{1, 2, \dots\}$. The *disjoint union* of a non-empty family $\{X_i\}_{i\in I}$ of topological spaces is the coproduct in the category of topological spaces and continuous functions and is denoted by $\bigsqcup_{i\in I} X_i$. If (X,p) and (Y,q) are pointed spaces, the *wedge sum* $X \wedge Y$ of (X,p) and (Y,q) is the quotient space of the disjoint union $X \sqcup Y$ of X and Y by the identification $p \sim q$. We shall use the notation: for a subset A of a vector space E and a natural number $n \in \mathbb{N}$ we denote by $\mathrm{sp}_n(A)$ the following subset of E

$$\mathrm{sp}_n(A) := \{\lambda_1 x_1 + \dots + \lambda_n x_n : \ \lambda_i \in [-n,n], \ x_i \in A, \ \forall i = 1, \dots, n\},$$

and set $\mathrm{sp}(A) := \bigcup_{n\in\mathbb{N}} \mathrm{sp}_n(A)$, the span of A in E.

As X is a Tychonoff space, the map i_X is an embedding. So we identify the space X with $i(X)$ and regard X as a subspace of $\mathbb{V}(X)$.

Theorem 2.3. *Let X be a Tychonoff space and $e \in X$ a distinguished point. Then*

(i) $\mathbb{V}(X)$ *and* $\mathbb{V}_G(X,e)$ *exist (and are Hausdorff);*
(ii) $\mathrm{sp}(X) = \mathbb{V}(X)$ *and X is a vector space basis for* $\mathbb{V}(X)$*;*
(iii) $\mathrm{sp}(X) = \mathbb{V}_G(X,e)$ *and $X \setminus \{e\}$ is a vector space basis for* $\mathbb{V}_G(X,e)$*;*
(iv) $\mathbb{V}(X)$ *and* $\mathbb{V}_G(X,e)$ *are unique up to isomorphism of topological vector spaces;*
(v) *X is a closed subspace of* $\mathbb{V}(X)$ *and* $\mathbb{V}_G(X,e)$*;*
(vi) $\mathrm{sp}_n(X)$ *is closed in* $\mathbb{V}(X)$ *and* $\mathbb{V}_G(X,e)$*, for every $n \in \mathbb{N}$;*
(vii) *if $q : X \to Y$ is a quotient map of Tychonoff spaces X and Y, then $\mathbb{V}(Y)$ is a quotient topological vector space of $\mathbb{V}(X)$;*
(viii) *if Y is a Tychonoff space with a distinguished point p and $X \wedge Y$ is the wedge sum of (X,e) and (Y,p), then* $\mathbb{V}_G(X,e) \times \mathbb{V}_G(Y,p) = \mathbb{V}_G\big(X \wedge Y, (e,p)\big)$.

Proof. (i) and (iv) follow from the Adjoint Functor Theorem.

(ii) and (iii) We consider only (ii) as (iii) is similarly proved. Let $x_1, x_2, \dots, x_n$ be distinct members of X and $\lambda_1, \lambda_2, \dots, \lambda_n$ non-zero members of $\mathbb{R}$ and put $v = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n \in \mathbb{V}(X)$. As X is a Tychonoff space, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x_1) = 1$ and $f(x_i) = 0$, for $i = 2, 3, \dots, n$. If $\bar{f}$ is the continuous linear map of $\mathbb{V}(X)$ into the topological vector space $\mathbb{R}$ of Definition 2.2, then $\bar{f}(v) = \lambda_1 \neq 0$. So $v \neq 0$ in $\mathbb{V}(X)$. Thus X is a vector space basis for $\mathbb{V}(X)$.