Topology and its Applications 221 (2017) 596–609

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On the coexistence of irreducible orbits of coincidences for multivalued admissible maps on the circle via Nielsen theory

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A R T I C L E   I N F O

Article history:
Received 31 March 2016
Received in revised form 21 November 2016
Accepted 15 February 2017
Available online 27 February 2017

MSC:
37E10
37E15
37C25
47H04
58C06

Keywords:
Coexistence
Irreducible orbits of coincidences
Admissible maps
Circles and tori
Nielsen theory
Degree

A B S T R A C T

The present paper can be regarded as a natural, significant, multivalued extension of some results matching the general periodic point theory (like the celebrated Sharkovsky-type theorems) and the standard Nielsen fixed point theory. Concretely, the coexistence of irreducible orbits of coincidences is established for multivalued circle maps by means of Nielsen-type topological invariants. A well known theorem for single-valued maps, obtained independently by Efremova [1] and Block et al. [2], is nontrivially generalized in this way. Some further possibilities for admissible maps on tori are indicated. Several illustrative examples are supplied. The crucial idea is based on detecting the kind of a complete isomorphism between periodic points of associated single-valued maps and irreducible orbits of coincidences of given multivalued admissible maps on tori.

1. Introduction

The main aim of the paper is to establish a multivalued analogy of the theorem, obtained independently by Efremova [1] and Block et al. [2], in the frame of the Nielsen fixed point theory. Their result (presented below as Theorem 3.2) deals with the coexistence of periodic points with various periods for continuous (single-valued) maps on the circle S^1.

Our theorem will be formulated neither in terms of periodic points, nor of periodic orbits, but of irreducible orbits of coincidences (for a justification, see Example 3.6 below) for multivalued maps which are

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[1] The author was supported by the grant No. 14-06958S "Singularities and impulses in boundary value problems for nonlinear ordinary differential equations" of the Grant Agency of the Czech Republic.

http://dx.doi.org/10.1016/j.topol.2017.02.071
0166-8641/

admissible in the sense of Górniewicz (see e.g. [3, Chapter I.4], [4, Chapter IV.40]). On S^1, these maps have, in particular, a closed graph and compact, connected, but not necessarily acyclic, set values. In general, this class is very large and contains acyclic maps, R_δ-maps, upper semicontinuous maps with compact, contractible values, and many others. It is the most natural class of multivalued maps for which the standard topological invariants like the generalized Lefschetz numbers, fixed point indices and degrees can be efficiently defined (see e.g. [4]). Moreover, they can be applied in a very powerful way to the needs of nonlinear analyses, multivalued dynamical systems and differential inclusions (see e.g. [3]).

The Nielsen theory, sometimes also called the theory of fixed point classes (cf. [5]), can be currently regarded as an almost synonym for the topological fixed point theory (cf. [6]). The Nielsen number, as the related topological invariant, estimates for single-valued maps from below the least number of fixed points which is preserved under a homotopy deformation.

As we will recall, for multivalued maps, the situation is quite different and must be essentially revised. Despite it, on tori, there fortunately exists a kind of isomorphism between periodic points of the associated single-valued maps and irreducible coincidences of admissible multivalued maps. Since this isomorphism is complete, to every admissible map on S^1 can be associated its degree, defined as the Lefschetz number plus 1. Subsequently, every admissible map on S^1 belongs to just one of the (admissible) homotopy classes treated in our main theorem. This topic is rather delicate and, as far as we know, there has not yet been detected any further analogy for different sorts of multivalued maps. It is a crucial point on which our paper relies. Furthermore, it also applies for higher-dimensional tori (see Theorem 5.3 below), but since the related apparatus is still more technical, we decided to restrict ourselves mainly to dimension 1, i.e. to circles, and only to indicate (as a flavour) the situation in higher dimensions in concluding remarks.

In fact, besides other things, admissible maps were also employed here to force the multivalued Nielsen theory, developed by ourselves in [7,8] (cf. also [3, Chapters I.10, II.6]), to be unified in the most natural way as possible, especially on circles, with a standard periodic point theory, as presented e.g. in [9].

In order our text to be sufficiently transparent and possibly readable for mathematicians who are not much familiar with the Nielsen theory, we had to avoid many technicalities, including the definitions of notions like homology, acyclicity, R_δ-set, (generalized) Lefschetz number, (H-) Nielsen number, Nielsen classes, Reidemeister classes, etc. All these notions can be found, jointly with the related theories, in the monographs [3,4].

In this light, we decided to concentrate ourselves mainly to just one well known theorem obtained independently by Efremova [1] and Block et al. [2] (cf. also [9, Theorem 3.4.1], where two alternative proofs are presented). We succeeded to get in Section 4 its full multivalued analogy for admissible circle self-maps (see Theorem 4.6 below). Some further possibilities are indicated in concluding remarks.

Our paper is organized as follows. In Preliminaries, we concentrated to admissible maps in the sense of Górniewicz, their compositions and admissible homotopies. In Elements of fixed point theory, we recalled the most important ingredients of the single-valued as well as multivalued Nielsen theory, based on our papers [7,8,10]. Section 4 is devoted to the main theorem and its corollary. Concluding remarks concern mainly possible extensions and open problems.

2. Preliminaries about admissible maps

In the entire text, all spaces under our consideration will be metric and all single-valued maps will be continuous. By a multivalued map $\varphi\colon X \multimap Y$, i.e. $\varphi\colon X \to 2^Y \setminus \{\emptyset\}$, we mean the one with at least nonempty closed values.

Definition 2.1. A map $\varphi\colon X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if, for every open $U \subset Y$, the set $\{x \in X : \varphi(x) \subset U\}$ is open in X, or equivalently, if for every closed $U \subset Y$, the set $\{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is closed in X.