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Chinese remainder approximation theorem

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ABSTRACT

We study a topological generalization of ideal co-maximality in topological rings and present some of its properties, including a generalization of the Chinese remainder theorem. Using the hyperspace uniformity, we prove a stronger version of this theorem concerning infinitely many ideals in supercomplete, pseudo-valuated rings. Finally we prove two interpolation theorems.

1. Introduction

This paper studies topological versions of the *Chinese remainder theorem – CRT* using two main concepts: topological co-maximality and the hyperspace uniformity. After establishing both of these notions, we proceed by proving the *Chinese remainder approximation theorem – CRAT* (Theorem 5.2). Our final results will be derived from it.

We begin by introducing the notion of *topological ideal co-maximality – TCM*, explaining what motivates our definition and presenting several examples. Then we show how some properties of co-maximality remain valid in the topological case. We also obtain a result resembling the second isomorphism theorem for co-maximal ideals in topological rings (Theorem 3.7). Finally, we prove a direct extension of the CRT for finite families of ideals (Theorem 3.8).

The hyperspace uniformity [5, p. 28] is used to study the case of infinite families of ideals. After a brief reminder of the basic definitions, we continue by studying topological co-maximality from the perspective

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of the hyperspace. From here on, our discussion is restricted to the class of pseudo-valuated rings. We show some approximation properties of those rings which will be used later to prove a strengthened version of the CRAT. Our main example is the ring of analytic functions over a domain in $\mathbb{C}$. This example will be used later to prove a statement about interpolation in infinite amount of points.

We then prove the CRAT for compact families of pairwise TCM ideals in general topological rings. Shortly afterwards, we present a stronger version for supercomplete, pseudo-valuated rings and provide some applications. In particular, we will prove two known interpolation theorems: [3, Corollary 9 on p. 366] and [8, Theorem 15.13].

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2. Preliminaries

All topological spaces mentioned below are Hausdorff. For every topological space X, we define $\mathrm{Conn}(X)$ to be the set of all connected components of X. Also, if A is a subset of X, then its closure is denoted by $\overline{A}$. The filter of neighborhoods at a given point $x \in X$ will be denoted by $\mathcal{N}_X(x)$ or simply $\mathcal{N}(x)$ when no confusion can arise. Any uniform space (X, μ) will be denoted by μX. If Y is a uniform space, then $C(X, Y)$ is the space of all continuous functions from X to Y with the uniformity of compact convergence.

If $(G, +)$ is an (abelian) topological group and ε is a neighborhood of the zero element, then $\frac{1}{n}\varepsilon$ is any neighborhood of zero such that

$$\underbrace{\frac{1}{n}\varepsilon + \cdots + \frac{1}{n}\varepsilon}_{\text{n times}} \subseteq \varepsilon.$$

All the rings contain an identity.

Let $\hat{\mathbb{C}}$ be the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Given a nonempty open set $\Omega \subset \hat{\mathbb{C}}$, we denote the topological ring of all analytic functions on Ω with the compact-open topology by $A(\Omega)$. By $\mathbb{N}$ we mean the set of all natural numbers including zero. Also, for any integer $n \in \mathbb{N}$, $f^{(n)}$ is the n'th derivative of f. More information on analytic functions can be found in [8].

3. Topological co-maximality

Two integers are said to be *co-prime* if their only common natural divisor is 1, or equivalently, if every integer can be written as a sum of their products. Similarly, two ideals I and J of a ring R are said to be *co-maximal* if there is no proper ideal containing them both, or equivalently, if $I + J = R$. We make a natural step generalizing this definition for topological rings. Instead of requiring $I + J$ to contain every element, we just want it to be dense in R.

Definition 3.1. Let R be a topological ring. We say that two ideals $I, J \trianglelefteq R$ are *topologically co-maximal (TCM)* and write $I \perp J$ if there is no *closed* proper ideal containing them both. Equivalently, this can be formulated as: $\overline{I + J} = R$.

We also say that a family of ideals $\mathcal{I}$ is *pairwise TCM* and write $\perp\mathcal{I}$ if any two distinct members are TCM.

If two ideals are co-maximal then they are topologically co-maximal. The converse is also true when R is discrete, but not in general.