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HL-homotopy of handlebody-links and Milnor's invariants ☆

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A handlebody-link is a disjoint union of embeddings of handlebodies in S^3 and an HL-homotopy is an equivalence relation on handlebody-links generated by self-crossing changes. The second author and Ryo Nikkuni classified the set of HL-homotopy classes of 2-component handlebody-links completely using the linking numbers for handlebody-links. In this paper, we construct a family of invariants for HL-homotopy classes of general handlebody-links, by using Milnor's $\overline{\mu}$-invariants. Moreover, we give a bijection between the set of HL-homotopy classes of almost trivial handlebody-links and tensor product space modulo some general linear actions, especially for 3- or more component handlebody-links. Through this bijection we construct invariants of HL-homotopy classes which can be used to distinguish the classes.

1. Introduction

A *link* is an embedding of some circles into the 3-sphere S^3. A *spatial graph* is a topological embedding of a graph in S^3. If all components of a spatial graph are homeomorphic to circles, the spatial graph is regarded as a link. Two spatial graphs are *equivalent* if there is an ambient isotopy which transform one to the other.

A *handlebody-link* [6,21] is a disjoint union of embeddings of handlebodies in S^3 (Fig. 1). Two handlebody-links are *equivalent* if there is an ambient isotopy which transforms one to the other. A handlebody-link can be represented by a spatial graph. A spatial graph G is said to *represent* a handlebody-link H if the regu-

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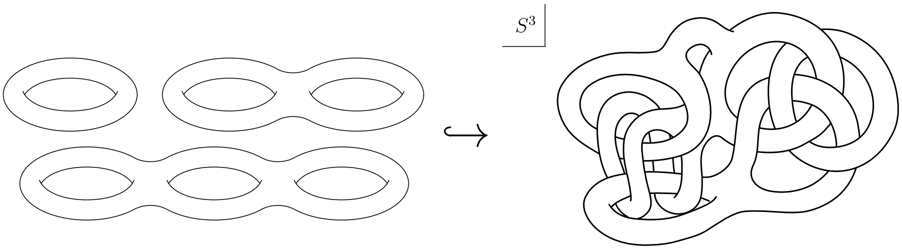

Fig. 1. Handlebody-link.

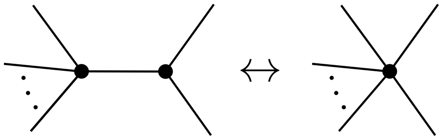

Fig. 2. Contraction move.

lar neighborhood of G is ambient isotopic to H. There are infinitely many spatial graphs which represent the same handlebody-link. It is known that two spatial graphs which represent the same handlebody-link are transformed to each other by a sequence of *contraction moves* in Fig. 2, which is to contract an edge connecting two different vertices and its inverse (see [6] for details). A handlebody-link is *trivial* if it is represented by a plane graph.

A *self-crossing change* of a link (resp. a spatial graph) is a crossing change of two arcs which belong to the same component of a link (resp. a spatial graph). J. Milnor defined a class of links called a *link-homotopy* [15]. Two links are *link-homotopic* if they are transformed to each other by self-crossing changes and ambient isotopies. The notion of link-homotopy was generalized for spatial graphs. An *edge-homotopy* [23] (resp. a *vertex-homotopy* [23], a *component-homotopy* [1]) is an equivalence relation generated by ambient isotopies and crossing changes of two arcs which belong to the same edge (resp. adjacent edges, the same component). We generalize the notion of link-homotopy to handleboody-links.

Definition 1.1 *(HL-homotopy).* Let H_0 be n handlebodies and H_i $(i = 1, 2)$ two n-component handlebody-links obtained by embeddings f_i's of H_0 to S^3. Two handlebody-links H_1 and H_2 are called *HL-homotopic* if there is homotopy h_t from f_1 to f_2 where the components of $h_t(H_0)$ are mutually disjoint at any $0 \leq t \leq 1$.

Remark 1.2. In [18], the notation of *neighborhood-homotopy* of spatial graphs was introduced. Two spatial graphs G and G' are neighborhood-homotopy if they are transformed to each other by a sequence of ambient isotopies, contraction moves and self-crossing changes. Let H_1 and H_2 be two handlebody-links and let G_1 and G_2 be spatial graph presentations of H_1 and H_2 respectively. The handlebody-links H_1 and H_2 are HL-homotopic if and only if G_1 and G_2 are neighborhood-homotopy.

J. Milnor classified the link-homotopy classes of 2-component links by linking numbers [15]. Moreover, he defined a family of invariants for an ordered oriented link in S^3 as a generalization of the linking numbers, in [15,16]. These invariants are called *Milnor's $\overline{\mu}$-invariants.* For an ordered oriented n-component link L, Milnor's $\overline{\mu}$-invariant is specified by a sequence I of indices in $\{1, 2, \ldots, n\}$ and denoted by $\overline{\mu}_L(I)$. If the sequence is with distinct indices, then this invariant is also link-homotopy invariant and called *Milnor's link-homotopy invariant.* He also classified the link-homotopy classes of 3-component links by the link-homotopy invariants $\overline{\mu}_L(I)$ with $|I| = 3$, where $|I|$ is the length of I. The link-homotopy classes of 4-component links were classified by [10]. In general, there is an algorithm which determines whether two link are link-homotopic or not [3].