Topology and its Applications 221 (2017) 15–19

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Dedicated to the 120th anniversary of the eminent Russian mathematician P.S. Alexandroff

Normal functors and hereditary paranormality

A.P. Kombarov

Mechanics-Mathematics Faculty, Moscow State University, 119991 Moscow, Russia

ARTICLE INFO

Article history:
Received 27 April 2016
Accepted 24 July 2016
Available online 2 March 2017

MSC:
54B20
54B30
54D15
54E35

Keywords:
Normal functor
Compact space
Hereditary paranormality
Metrizability

ABSTRACT

A topological space is said to be paranormal if every countable discrete collection of closed sets $\{D_n : n < \omega\}$ can be expanded to a locally finite collection of open sets $\{U_n : n < \omega\}$, i.e. $D_n \subset U_n$, and $D_m \cap U_n \neq \emptyset$ iff $D_m = D_n$. It is proved that if $\mathcal{F}$ is a normal functor $\mathcal{F} : Comp \to Comp$ of degree ≥ 3 and the space $\mathcal{F}(X) \setminus X$ is hereditarily paranormal, then the compact space X is metrizable.

Throughout this paper, all topological spaces are assumed to be regular. An ordinal is the set of smaller ordinals and a cardinal number is an initial ordinal. Other terminology and notations not defined in this paper can be found in [4].

P.S. Alexandroff and P.S. Urysohn [1] defined and studied the class of quasinormal spaces. A space is called quasinormal [1] or to have property D (in the sense of E.K. van Dowen [3]) if every countable closed discrete set has arbitrarily small closed neighborhoods. It is proved in ([3], Proposition 12.1) that the property of quasinormality is equivalent to the property that every countable discrete collection of singletons can be expanded to a discrete collection of open sets. A space is said to be paranormal (due to P. Nyikos [8]) if every countable discrete collection of closed sets $\{D_n : n < \omega\}$ can be expanded to a locally finite collection of open sets $\{U_n : n < \omega\}$, i.e. $D_n \subset U_n$, and $D_m \cap U_n \neq \emptyset$ iff $D_m = D_n$. Clearly, normal spaces and countably paracompact spaces are paranormal, and every paranormal space is quasinormal. A space X is hereditarily paranormal if every subspace of X is paranormal.

The well-known Tychonoff Plank $((\omega_1 + 1) \times (\omega + 1)) \setminus \{(\omega_1, \omega)\}$ is an example of a completely regular space without quasinormality.

E-mail address: apkombarov@gmail.com.

http://dx.doi.org/10.1016/j.topol.2017.02.054
0166-8641/

The next Lemma 1 contains the analogical example. Following [5], we denote by αN_τ (here τ is an infinite cardinal number) the Alexandroff compactification or the one-point compactification of a discrete set N_τ of cardinality τ.

Lemma 1. *If the cardinal number τ is uncountable, then the space $((\alpha N_\tau) \times (\omega + 1)) \setminus \{(\tau, \omega)\}$ is not a quasinormal space and hence is not a paranormal space.*

In 1989, Fedorchuk [5] proved the next Theorem 1.

Theorem 1. *If X is a compact space, $\mathcal{F}$ is a normal functor of degree ≥ 3, and the compact space $\mathcal{F}(X)$ is hereditarily normal, then X is metrizable.*

In 2000, Zhuraev [11] proved the Theorem 2.

Theorem 2. *If X is a compact space, $\mathcal{F}$ is a normal functor of degree ≥ 3, and the space $\mathcal{F}(X) \setminus X$ is hereditarily countably paracompact, then X is metrizable.*

Recall that a covariant functor $\mathcal{F} : Comp \to Comp$ acting in the category $Comp$ of compact spaces and their continuous mappings is normal if $\mathcal{F}$ has the following properties: the functor $\mathcal{F}$ is continuous, i.e. $\mathcal{F}$ is permutable with taking the limit of an inverse system; $\mathcal{F}$ preserves weight; $\mathcal{F}$ is monomorphic, i.e. it preserves the injectivity of mappings; $\mathcal{F}$ is epimorphic, i.e. $\mathcal{F}$ preserves the surjectivity of mappings; $\mathcal{F}$ preserves intersections, i.e. $\mathcal{F}(\cap\{F_\alpha : \alpha \in A\}) = \cap\{\mathcal{F}(F_\alpha) : \alpha \in A\}$; $\mathcal{F}$ preserves preimages; $\mathcal{F}$ preserves singletons and the empty set [9].

Suppose that X is a compact space, $\mathcal{F}$ is a normal functor, and $x \in \mathcal{F}(X)$. The degree of the point x is defined as the minimum positive integer n such that x belongs to the image $\mathcal{F}(f)$ for some mapping $f : K \to X$ of an n-point space K. If there exists no such a finite number n, then we assume that x has infinite degree. The degree of the functor $\mathcal{F}$ is the maximum degree of all points $x \in \mathcal{F}(X)$ for all compact spaces X [9]. For a normal functor $\mathcal{F}$ and for an arbitrary point $a \in \mathcal{F}(X)$, the support $supp(a)$ is defined as follows [9]:

$$supp(a) = \cap\{Y \subset X : a \in \mathcal{F}(Y)\}.$$

For any natural n, the subfunctor $\mathcal{F}_n$ of the functor $\mathcal{F}$ is defined by the formula

$$\mathcal{F}_n(X) = \{a \in \mathcal{F}(X) : |supp(a)| \leq n\}.$$

Since $\mathcal{F}_1(X)$ is homeomorphic to X, we can assume that the space X lies in $\mathcal{F}(X)$.

Obviously, all above-mentioned conditions are satisfied by the operation of raising a compact space X to the third power $(X \to X^3;\ (f : X \to Y) \to (f^3 : X^3 \to Y^3))$, and the degree of this functor equals three; thus, the theorem of Fedorchuk is a generalization of the well-known Katětov theorem on the cube [6]. Zhuraev [11] replaced the assumption of the hereditary normality of the compact space $\mathcal{F}(X)$ in the Fedorchuk theorem with the hereditary countable paracompactness of $\mathcal{F}(X) \setminus X$, by analogy with the Zenor theorem on the cube [10]. The following Theorem 3, is a simultaneous generalization of the theorems of Fedorchuk and Zhuraev. Its proof is given below.

Theorem 3. *If X is a compact space, $\mathcal{F}$ is a normal functor of degree ≥ 3, and the space $\mathcal{F}(X) \setminus X$ is hereditarily paranormal, then X is metrizable.*

The following Lemma 2 is proved in [5].