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# Topological properties preserved by weakly discontinuous maps and weak homeomorphisms

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A B S T R A C T

A map $f : X \to Y$ between topological spaces is called *weakly discontinuous* if each subspace $A \subset X$ contains an open dense subset $U \subset A$ such that the restriction $f|U$ is continuous. A bijective map $f : X \to Y$ between topological spaces is called a *weak homeomorphism* if $f$ and $f^{-1}$ are weakly discontinuous. We study properties of topological spaces preserved by weakly discontinuous maps and weak homeomorphisms. In particular, we show that weak homeomorphisms preserve network weight, hereditary Lindelöf number, dimension. Also we classify infinite zero-dimensional $\sigma$-Polish metrizable spaces up to a weak homeomorphism and prove that any such space $X$ is weakly homeomorphic to one of 9 spaces: $\omega$, $2^\omega$, $\mathbb{N}^\omega$, $\mathbb{Q}$, $\mathbb{Q} \oplus 2^\omega$, $\mathbb{Q} \times 2^\omega$, $\mathbb{Q} \oplus \mathbb{N}^\omega$, $(\mathbb{Q} \times 2^\omega) \oplus \mathbb{N}^\omega$, $\mathbb{Q} \times \mathbb{N}^\omega$.


## 1. Introduction

In this paper we detect topological properties preserved by weakly discontinuous maps and weak homeomorphisms.

By definition, a map $f : X \to Y$ between topological spaces is *weakly discontinuous* if each subspace $A \subset X$ contains an open dense subset $U \subset A$ such that the restriction $f|U$ is continuous. Such maps were introduced by Vinokurov [30] and studied in detail in [4–8,11–14,22,23]. Also they appear naturally in Analysis, see [9,16,19].

A bijective map $f : X \to Y$ between topological spaces is called a *weak homeomorphism* if $f$ and $f^{-1}$ are weakly discontinuous. In this case we say that the topological spaces $X$, $Y$ are *weakly homeomorphic*. In particular, we show that if $X$, $Y$ are weakly homeomorphic perfectly paracompact spaces, then

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(1) $nw(X) = nw(Y)$;
(2) $hd(X) = hd(Y)$;
(3) $\dim X = \dim Y$;
(4) $X$ is hereditarily Baire iff so is the space $Y$;
(5) $X$ is analytic iff so is the space $Y$;
(6) $X$ is $\sigma$-compact iff so is the space $Y$;
(7) $X$ is $\sigma$-Polish iff so is the space $Y$.

A topological space $X$ is called *$\sigma$-Polish* if it can be written as the countable union $X = \bigcup_{n\in\omega} X_n$ of closed Polish subspaces.

In Sections 2–4 we detect local and global properties of topological spaces, preserved by weakly discontinuous maps and weak homeomorphisms. In Section 5 we classify zero-dimensional $\sigma$-Polish spaces up to weak homeomorphism and prove that each infinite zero-dimensional $\sigma$-Polish metrizable space $X$ is weakly homeomorphic to one of 9 spaces: $\omega$, $2^\omega$, $\mathbb{N}^\omega$, $\mathbb{Q}$, $\mathbb{Q} \oplus 2^\omega$, $\mathbb{Q} \times 2^\omega$, $\mathbb{Q} \oplus \mathbb{N}^\omega$, $(\mathbb{Q} \times 2^\omega) \oplus \mathbb{N}^\omega$, $\mathbb{Q} \times \mathbb{N}^\omega$.

### 1.1. Terminology and notations

Our terminology and notation are standard and follow [3] and [18]. A "space" always means a "topological space". Maps between topological spaces are not necessarily continuous.

By $\mathbb{R}$ and $\mathbb{Q}$ we denote the spaces of real and rational numbers, respectively; $\omega$ stands for the space of finite ordinals (= non-negative integers) endowed with the discrete topology. The set $\omega \setminus \{0\}$ of finite positive ordinals (= natural numbers) is denoted by $\mathbb{N}$. We shall identify cardinals with the smallest ordinals of the given cardinality.

For a subset $A$ of a space $X$ by $\bar{A}$ we denote the closure of $A$ in $X$. For a function $f : X \to Y$ between topological spaces by $C(f)$ and $D(f) = X \setminus C(f)$ we denote the sets of continuity and discontinuity points of $f$, respectively.

Now we recall definitions of some cardinal invariants of topological spaces. For a topological space $X$

- its *network weight* $nw(X)$ is the smallest size $|\mathcal{N}|$ of a family $\mathcal{N}$ of subsets of $X$ such that for each point $x \in X$ and each neighborhood $U \subset X$ of $x$ there is a set $N \in \mathcal{N}$ such that $x \in N \subset U$;
- its *hereditary Lindelöf number* $hl(X)$ is the smallest cardinal $\kappa$ such that each open cover of a subspace $Y \subset X$ has a subcover of cardinality $\le \kappa$;
- its *hereditary density* $hd(X)$ is the smallest cardinal $\kappa$ such that each subspace $Y \subset X$ contains a dense subset of cardinality $\le \kappa$.

## 2. Topological properties, preserved by weakly discontinuous maps

In this section we discuss weakly discontinuous maps and detect topological properties preserved by such maps. We recall that a function $f : X \to Y$ between topological spaces is *weakly discontinuous* if any subspace $A \subset X$ contains an open dense subset $U \subset A$ such that the restriction $f|U$ is continuous. Observe that a function $f : X \to Y$ is weakly discontinuous if and only if every non-empty subspace $A \subset X$ contains a non-empty relatively open subset $V \subset A$ such that the restriction $f|V$ is continuous. This simple characterization implies the following useful fact, proved in Proposition 4.1 of [5].

**Lemma 2.1.** *For two weakly discontinuous functions $f : X \to Y$ and $g : Y \to Z$ between topological spaces $X$, $Y$, $Z$ the composition $g \circ f : X \to Z$ is weakly discontinuous.*