Topology and its Applications 221 (2017) 107–113

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# On metric order in spaces of the form $\mathcal{F}(X)$

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A R T I C L E   I N F O

*Article history:*
Received 14 January 2016
Accepted 9 October 2016
Available online xxxx

*MSC:*
54B20
54B30
54B99

*Keywords:*
Metric order
Capacitarian dimension
Normal functor
Superextension

A B S T R A C T

For a metrizable functor $\mathcal{F}$ and a point $\xi \in \mathcal{F}(Y)$ ($Y$ is a compact metric space) we define lower and upper metric orders $\underline{o}(\xi)$ and $\bar{o}(\xi)$ as a numerical characteristic of an approximation of $\xi$ by points $\xi_n \in \mathcal{F}_n(Y)$. If $\mathcal{F}$ is the exponential functor exp then $\underline{o}(\xi)$ and $\bar{o}(\xi)$ coincide, respectively, with classical lower and upper capacitarian dimensions $\underline{\dim}_B \xi$ and $\overline{\dim}_B \xi$ of a closed subset $\xi \subset Y$. We establish some properties of $\underline{o}(\xi)$ and $\bar{o}(\xi)$ and pose several questions.



Pontryagin and Shnirelman [1] defined the metric order for any closed subset $F$ of metric compact space $(Y, \rho)$ as a numerical characteristic for an approximation of $F$ by a sequence of finite subsets. In the first part of the paper we generalize this concept to any compact metric space $X$ with an increasing sequence of closed subsets $X_n$ such that the union $\cup X_n$ is dense in $X$. In such a way we define lower and upper metric orders $\underline{o}(q)$ and $\bar{o}(q)$ for any point $q \in X$.

In the second part we consider metric orders in the following special case. Let $\mathcal{F}$ be a metrizable functor in sense of V.V. Fedorchuk [2], let $(Y, \rho)$ be a compact metric space and let $\rho_{\mathcal{F}}$ be an extension of $\rho$ onto $\mathcal{F}(Y)$. We put $X = \mathcal{F}(Y)$, $X_n = \mathcal{F}_n(Y)$, $n \in N$ and consider upper and lower metric orders $\underline{o}^{\mathcal{F}}(\xi)$ and $\bar{o}^{\mathcal{F}}(\xi)$ for any point $\xi \in \mathcal{F}(Y)$. If $\mathcal{F}$ is the exponential functor exp then $\underline{o}^{\mathcal{F}}(\xi)$ and $\bar{o}^{\mathcal{F}}(\xi)$ coincide, respectively, with classical lower and upper capacitarian dimensions $\underline{\dim}_B \xi$ and $\overline{\dim}_B \xi$ of a closed subset $\xi$. We prove that

$$\underline{o}^{\exp \circ \exp}(\xi) = \underline{\dim}_B(\cup \xi), \bar{o}^{\exp \circ \exp}(\xi) = \overline{\dim}_B(\cup \xi),$$

for any $\xi \in \exp(exp(Y))$ and pose several questions, concerning, in particular, metric orders in a superextension $\lambda(Y)$.

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http://dx.doi.org/10.1016/j.topol.2017.02.051
0166-8641/

## 1. General definitions

Let $(X, \rho)$ be a metric space and $\{X_n : n \in \mathbf{N}\}$ is a sequence of closed subsets of $X$ such that $X_n \subset X_{n+1}$ and

$$Cl(\bigcup_{n\in\mathbf{N}} X_n) = X.$$

For each point $q \in X$ we define $E_n(q) = \rho(q, X_n)$. It is clear that

$$\lim_{n\to\infty} E_n(q) = 0$$

and $E_n(q) > 0$ for any $n$ if

$$q \notin \bigcup_{n\in\mathbf{N}} X_n.$$

The question of the rate of convergence of sequence $E_n(q)$ depending on properties of a point $q$ is the typical question in various theories of approximation.

It is possible to consider this question in another way. Let $\varepsilon > 0$ and $q \in X$. We set

$$N(q, \varepsilon) = \min\{n : \rho(q, X_n) \le \varepsilon\}.$$

If $q \notin \cup X_n$, then

$$\lim_{\varepsilon\to 0} N(q, \varepsilon) = \infty$$

and it is possible to study a rate of increasing of $N(q, \varepsilon)$ $(\varepsilon \to 0)$. First of all it is interesting, for which $\alpha > 0$ there exists lower limit

$$\underline{\lim}_{\varepsilon\to 0}\ \varepsilon^\alpha N(q, \varepsilon) = \underline{r}(q, \alpha)$$

or upper limit

$$\overline{\lim}_{\varepsilon\to 0}\ \varepsilon^\alpha N(q, \varepsilon) = \bar{r}(q, \alpha).$$

If $q \notin \cup X_n$ then $\underline{r}(q, 0) = \bar{r}(q, 0) = \infty$. If $\underline{r}(q, \alpha) = 0$ and $\alpha_1 > \alpha$ then $\underline{r}(q, \alpha_1) = 0$. The same is true for $\bar{r}(q, \alpha)$. We will define lower and upper metric order $\underline{o}(q)$ and $\bar{o}(q)$ of a point $q$ in the following way:

$$\underline{o}(q) = \inf\{\alpha : \underline{r}(q, \alpha) = 0\} = \sup\{\alpha : \underline{r}(q, \alpha) = \infty\},$$
$$\bar{o}(q) = \inf\{\alpha : \bar{r}(q, \alpha) = 0\} = \sup\{\alpha : \bar{r}(q, \alpha) = \infty\}.$$

It is clear that $0 \le \underline{o}(q) \le \bar{o}(q) \le \infty$. If $q \in \cup X_n$, then $\underline{o}(q) = \bar{o}(q) = 0$.

**Theorem 1.** *For any point* $q \in X$

$$\underline{o}(q) = \underline{\lim}_{\varepsilon\to 0} \frac{\ln(N(q, \varepsilon))}{\ln(\frac{1}{\varepsilon})}, \tag{1}$$

$$\bar{o}(q) = \overline{\lim}_{\varepsilon\to 0} \frac{\ln(N(q, \varepsilon))}{\ln(\frac{1}{\varepsilon})}. \tag{2}$$