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Restrictions of homeomorphisms of compactifications

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ABSTRACT

Conditions on equivalence of topological or uniform spaces having equivalent compactifications are investigated. For instance: (1) Let X, Y be Čech-complete, pseudoradial spaces complete in their compactifications bX, bY. If the compactifications are homeomorphic then X, Y are homeomorphic. (2) Let X, Y be products of complete uniform spaces having linearly ordered bases. If their Samuel compactifications are homeomorphic then X and Y are uniformly homeomorphic.

1. Introduction

P.S. Aleksandrov played an important role in development of homeomorphisms. G.H. Moore writes in his paper [15]: "The evolution of the concept of "homeomorphism" was essentially complete by 1935 when Pavel Aleksandrov at the University of Moscow and Heinz Hopf at the Eidgenossische Technische Hochschule in Zurich published their justly famous book Topologie, ...". They defined "A one-to-one continuous mapping f of a space X into a space Y is called a topological mapping or a homeomorphism (between X and $f(X) = Y^f \subset Y$) if the inverse of f is a continuous mapping of Y^f to X. Two spaces ... are called homeomorphic if they can each be mapped topologically onto each other".

From a modern point of view it is clear that isomorphisms in the category of topological spaces are homeomorphisms and it seems to be surprising that it took several decades than such a "trivial" fact was established. In fact, the term *homeomorphism* was used in various meanings in the second half of

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19-th century and that continued at the beginning of 20-th century. It was not quite clear whether a right role could not be played by two continuous one-to-one maps $X \to Y, Y \to X$. It was known from 1921 (see [14]) that existence of such maps does not imply existence of a homeomorphism. Hausdorff in his famous book *Grundzüge der Mengenlehre* does not mention homeomorphisms at all. He added a section on it in the new edition in 1927 since it started to be more evident that equivalence of topological spaces means homeomorphism as we know it today. Several characterizations (using homeomorphisms) of some basic spaces were already known by that time. P.S. Aleksandrov and P.S. Urysohn found conditions on a topological space to be homeomorphic to the space of irrationals in 1924 (the result was published in [1]), earlier such a characterization was given for rationals (Sierpinski) and Cantor set (Brouwer). About 1930 equivalence of topological spaces started to be used in the nowadays meaning.

In this paper, we shall be interested in the following situation. If two topological spaces are homeomorphic then their reflections in a subcategory are homeomorphic as well. For instance, two homeomorphic Tikhonov spaces have homeomorphic Čech–Stone compactifications. A natural question arises when a converse statement is true? We may ask a similar question for other compactifications, too. If bX is a compactification of X, then bX is a reflection in uniform spaces of X endowed with some convenient uniform structure (it is the Samuel compactification of X). And if the answer for a given pair of spaces is in the positive, is the homeomorphism between those spaces the restriction of the homeomorphism between the compactifications? Moreover, equivalence of spaces may be considered either in topological meaning or in a uniform sense or in a proximity sense. Our results generalize known results by Čech, Shirota, Isbell, Mrówka and of others.

Probably the first result of that kind belongs to Čech ([5], p. 835):

Theorem 1.1. *If X and Y are first-countable spaces, then the restriction to X of any homeomorphism on βX onto βY maps X onto Y.*

Clearly, the restriction is a homeomorphism. The result is a consequence of the next result by Čech (the same page of [5]) entailing that no point of $\beta X \setminus X$ is a G_δ-point in βX:

Theorem 1.2. *If $A \subset \beta X \setminus X$ is a nonvoid closed G_δ-set in βX, then $|A| \geq 2^{\omega_0}$.*

J. Novák mentions in his paper [17] that Čech announced at their seminar in 1939 the cardinality of any infinite closed subset of $\beta\mathbb{N}$ is 2^{2^ω}.

Some other compactifications can be used in Theorem 1.1 but certainly not arbitrary ones. An easy example shows that the result need not be true if one uses one-point compactifications of topologically complete locally compact metrizable spaces. For instance, take the subspaces $X = [0,1] \setminus \{1\}$ and $Y = [0,1] \setminus \{1/2\}$ of $\mathbb{R}$. Their one-point compactifications coincide with $[0,1]$ and the identity map $[0,1] \to [0,1]$ does not map X into Y.

A class of compactifications bX having the property that closed G_δ subsets A of bX disjoint with X are Čech–Stone remainders of $bX \setminus A$ is described in [11], Theorem 1.17 (also in [16], Theorem 3.7). Those compactifications have the property that if f is a function on X continuously extendable on bX with bounded inversion $1/f$, then $1/f$ can be continuously extended on bX, too. Then the set A contains a copy of $\beta\mathbb{N} \setminus \mathbb{N}$ and, consequently, has large cardinality. Thus, the remainders of those compactifications do not contain points with countable character, so that the Čech theorem about restrictions of homeomorphisms is valid for such compactifications as well.

We shall generalize the mentioned results both for more compactifications and for spaces with uncountable characters. Instead of large subsets of remainders we use convergence of filters or nets.

Uniformities can help in finding the requested results. Any compactification bX of a space X generates a totally bounded uniformity on X and its completion coincides with bX. For convenience of readers we recall some notations and properties of spaces used in the sequel. Standard references for topological and uniform spaces are the books [6] by Engelking and [12] by Isbell.