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Some properties of topological groups related to compactness

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A R T I C L E   I N F O

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To the memory of Academician Pavel Sergeevich Alexandroff

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A B S T R A C T

In the present article four problems from the A.V. Arhangel'skii and M.G. Tkachenko's book [8] are examined. Theorem 2.5 affirms that for any uncountable cardinal τ there exists a zero-dimensional hereditarily paracompact non-metrizable Abelian topological group G of the weight $\tau_1 = sup\{2^m : m < \tau\}$ which has a linearly ordered compactification bG of countable dyadicity index. In this connection, in Section 2 we present some properties of continuous images of Tychonoff product of compact spaces of the fixed weight τ. These spaces are called τ-dyadic. By virtue of Corollary 3.3, if G is a non-metrizable topological group of pointwise countable type, then the space $G_e = G \setminus \{e\}$ is not homeomorphic to a topological group. Section 3 contains also other results of that kind. In Section 4 some sufficient conditions are presented, under which the compact G_δ-subset of the quotient space G/H is a Dugundji space.

1. Introduction

In the present paper we continue the investigations from the articles [9–17,33,26,40]. By a space we understand a regular topological space. We use the terminology from [8,21]. Let ω denote the first infinite ordinal number and the first infinite cardinal number.

We will analyze four open problems formulated in [8] (Problems 1.4.1, 1.4.2, 1.5.1 and 4.7.8). In the connection with Problem 1.5.1, we present some properties of continuous images of Tychonoff product of compact spaces of the weight $\leq \tau$, where τ is a fixed cardinal number. These spaces are called τ-dyadic. The dyadic compact spaces were introduced by Pavel Sergeevich Alexandroff [1] as the continuous images of the Cantor cube D^m, where $D = \{0, 1\}$ is the discrete two-point space and m is an arbitrary cardinal number. The class of dyadic spaces was subsequently investigated by distinct mathematicians. Various subclasses of the class of dyadic spaces and different generalizations of the concept of dyadicity were discovered as

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a result (see [2,6,34,18–20,38,39]). Many important concepts and constructions arise in the theory of dyadic spaces in contact with the theory of topological groups. These constructions and concepts were applied in solving important problems proposed by P.S. Alexandroff, A.V. Arhangel'skii, A. Pelczynski, and other mathematicians.

The celebrated theorem of L.N. Ivanovskij and V.I. Kuz'minov [8, Theorem 4.1.7] solved one of the P.S. Alexandroff's problems and affirms that any compact topological group is a dyadic compactum. Later on, A. Pelczynski proved that any compact topological group is a Dugundji space [34]. The classes of Dugundji and Milutin spaces, as the subclasses of the class of dyadic spaces, were introduced for special constructions in the theory of topological function spaces [34]. A space X is called a Dugundji space [34] if it is compact and for any zero-dimensional compact space Z and every continuous mapping $f : Y \to X$, where Y is a closed subspace of Z, there exists a continuous mapping $g : Z \longrightarrow X$ extending f, i.e. $f = g|Y$. The R. Haydon's characteristic [25] of Dugundji spaces becomes an important research tool for that class of spaces.

2. Dyadicity indexes of spaces

Let $\mathbb{I} = [0, 1]$ be the unit closed interval in the usual topology. Let τ be an infinite cardinal number. The *dyadicity index* of a space X is $\leq \tau$ if there exists no continuous image of the space X onto the Tychonoff cube $\mathbb{I}^{\tau^+}$. The countable index of dyadicity was defined in [8, Problem 4.7.8]: The dyadicity index of a space X is countable if there exists no continuous image of the space X onto the Tychonoff cube $\mathbb{I}^{\aleph_1}$. If the dyadicity index of a space X is $> \tau$, then there exists a continuous mapping of the space X onto the Tychonoff cube $\mathbb{I}^{\tau}$. The dyadicity index of the space X is equal to the cardinal number $min\{\tau : \tau \textit{ is an infinite cardinal and there exists no continuous image of the space } X \textit{ onto } \mathbb{I}^{\tau^+}\}$.

Let τ be an infinite cardinal number. A space X is called a τ-dyadic space if X has a Hausdorff compactification bX which is a continuous image of a Tychonoff product of some family of compact spaces of the weight $\leq \tau$. Any dense subspace of a τ-dyadic space is a τ-dyadic space.

The R. Engelking's Theorem 8 in [22] can be reformulated in the following form:

Theorem 2.1. *If Y is a dense subspace of the τ-dyadic space X, then $w(X) \leq \tau + \chi(Y)$, where $\chi(Y) = sup\{\chi(y) : y \in Y\}$ is the character of the space Y.*

Let $D(\tau)$ be a discrete space of the cardinality τ. From Theorem 2.1 it follows that the Stone–Čech compactification $\beta D(\tau)$ of $D(\tau)$ is not an m-dyadic space for each $m < 2^{\tau}$.

The smallest infinite cardinal number τ such that every family of cardinality $> \tau$ consisting of non-empty open subsets of X contains a subfamily of cardinality $> \tau$ with non-empty intersection is called the Shanin number and it is denoted by $s(X)$. The Souslin number or the cellularity $c(X)$ of a space X is the smallest cardinal number τ such that every family of pairwise disjoint non-empty open subsets of X has cardinality $\leq \tau$. It is well known that $c(X) < s(X) \leq d(X)^+$, where $d(X)$ is the density of the space X and τ^+ is the first infinite cardinal greater than τ (see [21,36]). From the N.A. Shanin's Theorem in [36] (see [21, Problem 2.7.11]) it follows:

Corollary 2.2. *For any τ-dyadic space X we have $c(X) \leq \tau$ and $s(X) = \tau^+$.*

A space X is said to be pseudo-τ-compact if every discrete in X family of open non-empty sets has cardinality strictly less than τ [8]. From Proposition 1.6.22 from [8] it follows.

Corollary 2.3. *Let τ be an infinite cardinal number. Then any compact τ-dyadic space X is pseudo-τ^+-compact.*