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The locked cohomology of the torus ☆

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We calculate the ring of the locked cohomology of the torus.

1. Introduction

For a semi-ideal $\mathcal{F} \subset \mathrm{Conj}_G$ of orbit types, a universal $\mathcal{F}$-space $\mathbb{E}_{\mathcal{F}}$ (in the sense of tom Dieck [1]) generates an equivariant cohomology theories $H^*_{\mathcal{F}}$ (called the tom Dieck cohomology theories) which assigns the ring $H^*_{\mathcal{F}}(\mathbb{X}) = H^*((\mathbb{E}_{\mathcal{F}} \times \mathbb{X})/G; \mathbb{Q})$ of the Čech cohomology to $\mathbb{X}$ (here $\mathbb{Q}$ is the field of rational numbers). A partial case of $H^*_{\mathcal{F}}$ is the equivariant Borel cohomology corresponding $\mathcal{F} = \{(e)\}$.

The tom Dieck cohomology possesses many convenient formal properties, but its computation is difficult (in comparison with the classical Borel equivariant cohomology). However, within the framework of the theory of the universal Palais G-spaces (in other words, the theory of isovariant extensors, or Isov-AE-spaces, see [2]), there appears a new possibility for construction and computation of the tom Dieck cohomology. It turns out that universal $\mathcal{F}$-spaces exist for all families $\mathcal{F}$ of orbit types, and all of them are concentrated within an Isov-AE-space $\mathbb{W}$ (the so-called effect of concentration): $\mathbb{W}_{\mathcal{F}}$ is a $\mathcal{F}$-universal space in the sense

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of tom Dieck. Moreover, they have the properties of mutual location in $\mathbb{W}$ as good as it can be possible. This method allows to bring the computations of some cohomology invariants to the explicit formulae.

Since $H^*_{\mathcal{F}}(\mathbb{X})$ is an algebra over the ring $H^*_{\mathcal{F}}(*) = \check{H}^*(W_{\mathcal{F}} = \mathbb{W}_{\mathcal{F}}/G; \mathbb{Q})$, we have a strong need for its calculation. In general, this problem seems to be quite difficult. Even for a compact connected Lie group G, the problem of computation of the cohomology $H^*_{\mathcal{P}}(*)$ where $\mathcal{P} = \mathrm{Conj}_G \setminus \{(e)\}$ (which we prefer to call a *locked cohomology*) is still far from being completely solved, though it presents an considerable interest in the homogeneity conjecture for the Banach–Mazur compactum (in particular, the establishment of its non-triviality) (see [3]). It is known that $H^*_{\mathcal{P}}(*)$ is the polynomial ring $\mathbb{Q}[x]$ with generator x of degree 2 (for $G = S^1$) and 4 (for $G = O(2)$) (see [2]). The aim of the paper is the computation of the locked cohomology for $G = \mathrm{T}^2 = S^1 \times S^1$.

Theorem 1. $\check{H}^q(W_{\mathcal{P}}, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \textit{if } q = 0 \\ 0, & \textit{if } q = 1 \textit{ or } q = 2k,\ k \in \mathbb{N} \\ \mathbb{Q}^\omega, & \textit{if } q = 2k+1,\ k \in \mathbb{N}, \end{cases}$
where $\mathbb{Q}^\omega$ denotes the countable power of $\mathbb{Q}$.

Theorem 1 implies that the multiplication in $H^*_{\mathcal{P}}(*)$ is trivial.

2. Preliminary facts and results

In what follows we shall assume all spaces (all maps) to be metric (continuous, respectively), if they do not arise as a result of some constructions or if the opposite is not claimed; all acting groups are assumed to be compact Lie groups.

We present the basic notions of the theory of G-spaces [4]. An action of a compact group G on a space $\mathbb{X}$ is a continuous map μ from the product $G \times \mathbb{X}$ into $\mathbb{X}$ satisfying the following properties:

1. $\mu(g, \mu(h, x)) = \mu(g \cdot h, x)$; and
2. $\mu(e, x) = x$ for all $x \in \mathbb{X}$, $g, h \in G$ (here e is the unit of the group G).

As a rule, $\mu(g, x)$ will be written as $g \cdot x$ or just gx. A space $\mathbb{X}$ with an action of the group G is called a *G-space*. The map $f\colon \mathbb{X} \to \mathbb{Y}$ of G-spaces is called a *G-map* or an *equivariant map* if $f(g \cdot x) = g \cdot f(x)$ for all $x \in \mathbb{X}$, $g \in G$.

The subset $\{g \cdot x \mid g \in G\} = G \cdot x$ is called the *orbit* $G(x)$ of the point $x \in \mathbb{X}$ which turns out to be closed. The natural map $\pi = \pi_{\mathbb{X}}\colon \mathbb{X} \to \mathbb{X}/G$, $x \mapsto G(x)$, of the space $\mathbb{X}$ onto the space $\mathbb{X}/G$ of quotient partition is said to be the *orbit projection*. We call the space of quotient partition, equipped with the quotient topology induced by π, the *orbit space*. We will denote it by $X \rightleftharpoons \mathbb{X}/G$, provided that no confusion occurs. For each point $x \in \mathbb{X}$ the subset $G_x = \{g \in G \mid g \cdot x = x\}$ is a closed subgroup of G and is called a *stabilizer of* x.

For each closed subgroup $H < G$ let us consider the following subsets $\mathbb{X}$:

$$\mathbb{X}^H = \{x \in \mathbb{X} \mid H \cdot x = x\} = \{x \in \mathbb{X} \mid H \subset G_x\}$$

(*the set of H-fixed points*),

$$\mathbb{X}_H = \{x \in \mathbb{X} \mid H = G_x\}, \qquad \mathbb{X}_{(H)} = \{x \in \mathbb{X} \mid H \text{ conjugates with } G_x\}.$$

In particular when H is the trivial subgroup $\{e\}$ or $H = G$, we have:

$$\mathbb{X}_{free} = \mathbb{X}_e = \{x \in \mathbb{X} \mid G_x = \{e\}\}, \qquad \mathbb{X}_{fix} = \mathbb{X}_G = \{x \in \mathbb{X} \mid G_x = G\}.$$