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On homology of complements of compact sets in Hilbert cube

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A B S T R A C T

We introduce the notion of cohomologically weakly infinite-dimensional spaces and show the acyclicity of the complement $Q \setminus X$ in the Hilbert cube Q of a cohomologically weakly infinite-dimensional compactum X. As a corollary we obtain the acyclicity of the complement results when

(a) X is weakly infinite-dimensional;
(b) X has finite cohomological dimension.

1. Introduction

This paper was motivated in part by a recent results of Belegradek and Hu [5] about connectedness properties of the space $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ of non-negative curvature metrics on $\mathbb{R}^2$, where k stands for C^k topology. In particular, they proved that the complement $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2) \setminus X$ is connected for every finite-dimensional X. In [2] this result was extended to weakly infinite-dimensional spaces X. The main topological idea of these results is that a weakly infinite-dimensional compact set cannot separate the Hilbert cube.

Banakh and Zarichnyi posted in [10] a problem (Problem Q1053) whether the complement $Q \setminus X$ to a weakly infinite-dimensional compactum X in the Hilbert cube Q is acyclic. They were motivated by the facts that $Q \setminus X$ is acyclic in the case when X is finite-dimensional [9] and when it is countable dimensional (see [4]). It turns out that the problem was already answered affirmatively by Garity and Wright in the 80s [8]. It follows from Theorem 4.5 [8], which states that finite codimension closed subsets of the Hilbert cube are strongly infinite-dimensional. In this paper we found some sufficient cohomological conditions on compact metric spaces for the acyclicity of the complement in the Hilbert cube which gives an alternative solution for the Banakh–Zarichnyi problem.

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We introduced a cohomological version of the concept of strongly infinite-dimensional spaces. We call spaces which is not cohomologically strongly infinite-dimensional as *cohomologically weakly infinite-dimensional spaces*. The main result of the paper is an acyclicity statement for the complement of cohomologically weakly infinite-dimensional compacta in the Hilbert cube.

2. Main theorem

2.1. Alexander duality

We recall that for a compact set $X \subset S^n$ there is a natural isomorphism $AD_n : H^{n-k-1}(X) \to \tilde{H}_k(S^n \setminus X)$ called the Alexander duality. The naturality means that for a closed subset $Y \subset X$ there is a commutative diagram (1):

$$\begin{array}{ccc} H^{n-k-1}(X) & \xrightarrow{AD_n} & \tilde{H}_k(S^n \setminus X) \\ \downarrow & & \downarrow \\ H^{n-k-1}(Y) & \xrightarrow{AD_n} & \tilde{H}_k(S^n \setminus Y). \end{array}$$

Here we use the singular homology groups and the Čech cohomology groups. We note the Alexander Duality commutes with the suspension isomorphism s in the diagram (2):

$$\begin{array}{ccc} H^{n-k-1}(X) & \xrightarrow{AD_n} & \tilde{H}_k(S^n \setminus X) \\ {\scriptstyle s}\downarrow & & \cong\downarrow \\ H^{n-k}(\Sigma X) & \xrightarrow{AD_{n+1}} & \tilde{H}_k(S^{n+1} \setminus \Sigma X). \end{array}$$

This diagram can be viewed as a special case of the Spanier–Whitehead duality [11].

Since $B^n/\partial B^n = S^n$, the Alexander Duality in the n-sphere S^n and its property can be restated verbatim for the n-ball B^n and subspaces $X \subset B^n$ with $X \cap \partial B^n \neq \emptyset$ in terms of relative cohomology groups

$$H^{n-k-1}(X, X \cap \partial B^n) \xrightarrow{AD} \tilde{H}_k(\operatorname{Int} B^n \setminus X).$$

Note that for a pointed space X the reduced suspension ΣX is the quotient space $(X \times I)/(X \times \partial I \cup x_0 \times I)$ and the suspension isomorphism $s : H^i(X, x_0) \to H^{i+1}(\Sigma X) = H^{i+1}(X \times I, X \times \partial I \cup x_0 \times I)$ is defined by a cross product with the fundamental class $\phi \in H^1(I, \partial I)$, $s(\alpha) = \alpha \times \phi$. Note that $\alpha \times \phi = \alpha^* \cup \phi^*$ in $H^{i+1}(X \times I, X \times \partial I \cup x_0 \times I)$, where $\alpha^* \in H^i(X \times I, x_0 \times I)$ is the image of α under the induced homomorphism for the projection $X \times I \to X$ and $\phi^* \in H^1(X \times I, X \times \partial I)$ is the image of ϕ under the induced homomorphism defined by the projection $X \times I \to I$. Thus, in view of the isomorphism $\tilde{H}_*(\operatorname{Int} B^n \setminus X) \to \tilde{H}_*(B^n \setminus X)$, the commutative diagram (2) stated for B^n turns into the following (2′)

$$\begin{array}{ccc} H^{n-k-1}(X, \partial X) & \xrightarrow{AD_n} & \tilde{H}_k(B^n \setminus X) \\ {\scriptstyle -\cup\phi^*}\downarrow & & \cong\downarrow \\ H^{n-k}(X \times I, \partial X \times I \cup X \times \partial I) & \xrightarrow{AD_{n+1}} & \tilde{H}_k(B^{n+1} \setminus (X \times I)) \end{array}$$

where $\partial X = X \cap \partial I^n$.