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On sequential separability of functional spaces

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ABSTRACT

In this paper, we give necessary and sufficient conditions for the space $B_1(X)$ of first Baire class functions on a Tychonoff space X, with pointwise topology, to be (strongly) sequentially separable. Also we claim that there are spaces X such that $B_1(X)$ is not sequentially separable space, but $C_p(X)$ is sequentially separable (the Sorgenfrey line, the Niemytzki plane).

1. Introduction

In [2,3] were given necessary and sufficient conditions for the space $C_p(X)$ of continuous real-valued functions on a space X, with pointwise topology, to be sequentially separable. Also in [3] was given necessary and sufficient condition for the space $C_p(X)$ to be strongly sequentially separable.

In this paper, we give necessary and sufficient conditions for the space $B_1(X)$ of first Baire class functions on a space X, with pointwise topology, to be sequentially separable and strongly sequentially separable.

2. Main definitions and notation

Throughout this article all topological spaces are considered Tychonoff. As usually, we will be denoted by $C_p(X)$ ($B_1(X)$) a set of all real-valued continuous functions $C(X)$ (a set of all first Baire class functions

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$B_1(X)$ i.e., pointwise limits of continuous functions) defined on X provided with the pointwise convergence topology. If X is a space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. If D is a countable sequentially dense subset of X then X call sequentially separable space.

Call X strongly sequentially separable if X is separable and every countable dense subset of X is sequentially dense.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set. If a set $Z = \bigcup_i Z_i$ where Z_i is a zero-set of X for any $i \in \omega$ then Z is called Z_σ-set of X. Note that if a space X is a perfect normal space, then class of Z_σ-sets of X coincides with class of F_σ-sets of X.

It is well known [5], that $f \in B_1(X)$ if and only if $f^{-1}(G)$ — Z_σ-set for any open set G of real line $\mathbb{R}$.

Further we use the following theorems.

Theorem 2.1. *([2]). A space $C_p(X)$ is sequentially separable if and only if there exist a condensation (one-to-one continuous map) $f : X \mapsto Y$ from a space X on a separable metric space Y, such that $f(U)$ — F_σ-set of Y for any cozero-set U of X.*

Theorem 2.2. *([2]). A space $B_1(X)$ is sequentially separable for any separable metric space X.*

Note that proof of this theorem gives more, namely there exists a countable subset $S \subset C(X)$, such that $[S]_{seq} = B_1(X)$.

3. Sequentially separable of $B_1(X)$

The main result of this paper is a next theorem.

Theorem 3.1. *A space $B_1(X)$ is sequentially separable if and only if there exists a bijection $\varphi : X \mapsto Y$ from a space X onto a separable metrizable space Y, such that*

1. *$\varphi^{-1}(U)$ — Z_σ-set of X for any open set U of Y;*
2. *$\varphi(T)$ — F_σ-set of Y for any zero-set T of X.*

Proof. (1) ⇒ (2). Let $B_1(X)$ be a sequentially separable space, and S be a countable sequentially dense subset of $B_1(X)$. Consider a topology τ generated by the family $\mathcal{P} = \{f^{-1}(G) : G$ is an open set of $\mathbb{R}$ and $f \in S\}$. A space $Y = (X, \tau)$ is a separable metrizable space because S is a countable dense subset of $B_1(X)$. Note that a function $f \in S$, considered as map from Y to $\mathbb{R}$, is a continuous function. Let φ be the identity map from X on Y.

We claim that $\varphi^{-1}(U)$ — Z_σ-set of X for any open set U of Y. Note that class of Z_σ-sets is closed under a countable unions and finite intersections of its elements. It follows that it is sufficient to prove for any $P \in \mathcal{P}$. But $\varphi^{-1}(P)$ — Z_σ-set for any $P \in \mathcal{P}$ because $f \in S \subset B_1(X)$.

Let T be a zero-set of X and h be a characteristic function of T. Since T is a zero-set of X, $h \in B_1(X)$. There are $\{f_n\}_{n\in\omega} \subset S$ such that $\{f_n\}_{n\in\omega}$ converges to h. Since $S \subset C_p(Y)$, $h \in B_1(Y)$ and, hence, $h^{-1}(\frac{1}{2}, \frac{3}{2}) = T$ is a Z_σ-set of Y.

(2) ⇒ (1). Let φ be a bijection from X on Y satisfying the conditions of theorem. Then $h = f \circ \varphi \in B_1(X)$ for any $f \in C(Y)$ ($h^{-1}(G) = \varphi^{-1}(f^{-1}(G))$ — Z_σ-set of X for any open set G of $\mathbb{R}$). Moreover $g = f \circ \varphi^{-1} \in B_1(Y)$ for any $f \in B_1(X)$ because of $\varphi(Z)$ is a Z_σ-set of Y for any a Z_σ-set Z of X. Define a map $F : B_1(X) \mapsto B_1(Y)$ by $F(f) = f \circ \varphi^{-1}$. Since φ is a bijection, $C(Y)$ embeds in $F(B_1(X))$ i.e., $C(Y) \subset F(B_1(X))$. By Theorem 2.2, each subspace D such that $C(Y) \subset D \subset B_1(Y)$ is sequentially separable. Thus $B_1(X)$ (homeomorphic to $F(B_1(X))$) is sequentially separable. □