Topology and its Applications 221 (2017) 286–299

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Dedicated to the 120th anniversary of the eminent Russian mathematician P.S. Alexandroff

Hurewicz-type properties in texture structures

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ARTICLE INFO

Article history:
Received 16 May 2016
Accepted 8 September 2016
Available online 14 February 2017

MSC:
primary 54D20
secondary 05C55, 54A05, 54H12, 91A44

Keywords:
Texture space
Ditopological texture space
Hurewicz property
Di-uniformity
Games
Ramsey theory

ABSTRACT

We continue our study of selection principles in texture and ditopological texture spaces began in [14]. Our focus here is on the Hurewicz covering properties in texture structures and direlational uniform texture spaces. The behavior of these properties under standard operations with ditopological spaces are considered. We give game-theoretic and Ramsey-theoretic characterizations of the Hurewicz property in texture spaces, and establish relationships between selection properties of di-uniform spaces and (quasi-) uniform spaces.

1. Introduction

The theory of selection principles has a long history going back to 1920s and 1930s. However, classical selection principles have been defined in a general form in [24] as follows.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consist of families of subsets of an infinite set X. Then:

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$: For each sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{A}$ there is a sequence $(b_n)_{n\in\mathbb{N}}$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$: For each sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{A}$ there is a sequence $(B_n)_{n\in\mathbb{N}}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

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http://dx.doi.org/10.1016/j.topol.2017.02.053
0166-8641/

If $\mathcal{O}$ denotes the family of all open covers of a topological space X, then X has the *Rothberger property* [22] (resp. the *Menger property* [16]) if X satisfies $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ (resp. $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$).

In [14] we studied the Menger and Rothberger properties in texture structures. In this paper we investigate properties of texture structures related to the following *Hurewicz property* [7]: For each sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of a space X there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\cup\mathcal{V}_n = \cup\{V : V \in \mathcal{V}_n\}$ for all but finitely many n. It was shown in [15] that the Hurewicz property is of the S_{fin}-type for appropriate classes $\mathcal{A}$ and $\mathcal{B}$.

For more information about selection principles (and the corresponded games) in topological spaces see [10,23–25], in bitopological spaces the papers [12,13], and in uniform structures [9,11].

Texture spaces: ([1]) Let S be a set and $\mathcal{P}(S)$ its power set. A *texturing* of S is a lattice $(\mathcal{S}, \subseteq) \subseteq \mathcal{P}(S)$ which is point-separating (i.e. for distinct x, y in S there is an $A \in \mathcal{S}$ such that $x \in A$ and $y \notin A$, or $y \in A$ and $x \notin A$), complete, completely distributive, contains S and $\emptyset$, and for which arbitrary meets coincide with intersections, and finite joins with unions. If $\mathcal{S}$ is a texturing of S, the pair $(S, \mathcal{S})$ is called a *texture*. Throughout the paper we denote by $\bigcap$ and $\bigvee$ meets and joins in a texture $(S, \mathcal{S})$.

For $s \in S$ the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\} \text{ and } Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$$

are called respectively, the *p-sets* and *q-sets* of $(S, \mathcal{S})$. These sets are used in the definition of many textural concepts.

In a texture, arbitrary joins need not coincide with unions, and clearly, this will be so if and only if $\mathcal{S}$ is closed under arbitrary unions, or equivalently if $P_s \not\subseteq Q_s$ for all $s \in S$. In this case $(S, \mathcal{S})$ is said to be *plain*.

A mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$, $\forall\, A \in \mathcal{S}$ and $A \subseteq B \Longrightarrow \sigma(B) \subseteq \sigma(A)$, $\forall\, A, B \in \mathcal{S}$ is called a *complementation* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is then said to be a *complemented texture* ([1])

If $\mathcal{F}$ is a subfamily of $\mathcal{S}$, then $\sigma(\mathcal{F})$ denotes the set $\{\sigma(F) : F \in \mathcal{F}\}$.

Examples:

1. For any set X, $(X, \mathcal{P}(X), \pi_X)$ is the complemented *discrete texture* representing the usual set structure of X. Here the complementation $\pi_X(Y) = X \setminus Y$, $Y \subseteq X$, is the usual set complement. Clearly, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$.

2. For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] : t \in [0, 1]\} \cup \{[0, t) : t \in [0, 1]\}$. $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in I$.

3. The texture $(\mathbb{L}, \mathcal{L})$ is defined by $\mathbb{L} = (0, 1]$ and $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$. For $r \in \mathbb{L}$ $P_r = (0, r] = Q_r$.

Ditopology: A *dichotomous topology* on $(S, \mathcal{S})$ or *ditopology* for short, is a pair (τ, κ) of generally unrelated subsets τ, κ of $\mathcal{S}$ satisfying

(τ_1): $S, \emptyset \in \tau$,
(τ_2): $G_1, G_2 \in \tau \Longrightarrow G_1 \cap G_2 \in \tau$,
(τ_3): $G_i \in \tau$, $i \in I \Longrightarrow \bigvee_i G_i \in \tau$,
(κ_1): $S, \emptyset \in \kappa$,
(κ_2): $K_1, K_2 \in \kappa \Longrightarrow K_1 \cup K_2 \in \kappa$,
(κ_3): $K_i \in \kappa$, $i \in I \Longrightarrow \bigcap K_i \in \kappa$.

The elements of τ are called *open* and those of κ *closed*. We refer to τ as the *topology* and κ as the *cotopology* of (τ, κ).

If (τ, κ) is a ditopology on a complemented texture $(S, \mathcal{S}, \sigma)$, then we say that (τ, κ) is *complemented* if the equality $\kappa = \sigma[\tau]$ is satisfied. A complemented ditopological texture space is denoted by $(S, \mathcal{S}, \sigma, \tau, \kappa)$.