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# Local structure of Gromov–Hausdorff space, and isometric embeddings of finite metric spaces into this space

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We investigate the geometry of the family $\mathcal{M}$ of isometry classes of compact metric spaces, endowed with the Gromov–Hausdorff metric. We show that sufficiently small neighborhoods of generic finite spaces in the subspace of all finite metric spaces with the same number of points are isometric to some neighborhoods in the space $\mathbb{R}^N_\infty$, i.e., in the space $\mathbb{R}^N$ with the norm $\|(x_1,\dots,x_N)\| = \max_i |x_i|$. As a corollary, we get that each finite metric space can be isometrically embedded into $\mathcal{M}$ in such a way that its image belongs to a subspace consisting of all finite metric spaces with the same number $k$ of points. If the initial space has $n$ points, then one can take $k$ as the least possible integer with $n \le k(k-1)/2$.



## 1. Introduction

By $\mathcal{M}$ we denote the space of all compact metric spaces (considered up to an isometry) endowed with the Gromov–Hausdorff metric. It is well-known that $\mathcal{M}$ is linear connected, complete, separable, but not proper. In a recent paper [1], A. Ivanov, N. Nikolaeva, and A. Tuzhilin have shown that $\mathcal{M}$ is geodesic. There are many other open questions concerning geometrical properties of $\mathcal{M}$. S. Iliadis formulated the following problems.

**Problem (1):** *is it true that $\mathcal{M}$ contains isometrically any compact metric space, in particular, any finite space?*

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**Problem (2):** *suppose that $K$ is a compact metric space, $L \subset K$ is its compact subspace, and there exist isometric embeddings of $L$ and $K$ into $\mathcal{M}$. Is it true that each isometric embedding $L \to \mathcal{M}$ can be extended to an isometric embedding $K \to \mathcal{M}$?*

It is easy to verify that the answer to Problem (2) is negative (see Section 3). Concerning Problem (1), we show the following: *each finite metric space can be isometrically embedded into $\mathcal{M}$*. Moreover, we construct such an embedding with its image belonging to the subspace of all finite metric spaces with $k$ points: if the initial space has $n$ points, then one can choose $k$ as the least possible integer such that $n \le k(k-1)/2$.

The construction of such embedding is based on our results concerning the local geometry of the family of finite metric spaces with fixed number of points considered in sufficiently small neighborhoods of generic spaces. More precisely, we show that such neighborhoods are isometric to some neighborhoods of the corresponding points in the space $\mathbb{R}^k_\infty$, i.e., in the space $\mathbb{R}^k$ with the norm $\big\|(x_1,\dots,x_k)\big\| = \max_i |x_i|$.

## 2. Preliminaries

Let $X$ be an arbitrary metric space. By $|xy|$ we denote the distance between points $x$ and $y$ in $X$. For every point $x \in X$ and a real number $r > 0$ by $U_r(x)$ we denote the open ball of radius $r$ centered at $x$; for every nonempty $A \subset X$ and real number $r > 0$ we put $U_r(A) = \cup_{a\in A} U_r(a)$.

For nonempty $A$, $B \subset X$, let us put

$$d_H(A,B) = \inf\big\{r > 0 : A \subset U_r(B) \& B \subset U_r(A)\big\}.$$

This value is called the *Hausdorff distance between $A$ and $B$*. It is well-known [2] that the restriction of the Hausdorff distance to the family of all closed bounded subsets of $X$ is a metric.

Let $X$ and $Y$ be metric spaces. A triple $(X', Y', Z)$ that consists of a metric space $Z$ and its subsets $X'$ and $Y'$ isometric to $X$ and $Y$, respectively, is called a *realization of the pair* $(X,Y)$. The *Gromov–Hausdorff distance $d_{GH}(X,Y)$ between $X$ and $Y$* is the greatest lower bound of the real numbers $r$ such that there exists a realization $(X', Y', Z)$ of the pair $(X,Y)$ with $d_H(X', Y') \le r$. It is well-known [2] that the $d_{GH}$ restricted to the family $\mathcal{M}$ of isometry classes of compact metric spaces is a metric.

Recall that a *relation* between sets $X$ and $Y$ is a subset of the Cartesian product $X \times Y$. By $\mathcal{P}(X,Y)$ we denote the set of all nonempty relations between $X$ and $Y$. If $\pi_X : (x,y) \mapsto x$ and $\pi_Y : (x,y) \mapsto y$ are the canonical projections, then their restrictions to each $\sigma \in \mathcal{P}(X,Y)$ are denoted in the same manner.

We consider each relation $\sigma \in \mathcal{P}(X,Y)$ as a multivalued mapping, whose domain may be less than the whole $X$. By analogy with mappings, for every $x \in X$ its image $\sigma(x) = \{y \in Y \mid (x,y) \in \sigma\}$ is defined, and for every $y \in Y$ its preimage $\sigma^{-1}(y) = \{x \in X \mid (x,y) \in \sigma\}$ is defined also; for every $A \subset X$ its image $\sigma(A)$ is the union of the images of all the elements from $A$, and, similarly, for every $B \subset Y$ its preimage is the union of the preimages of all the elements from $B$.

A relation $R$ between $X$ and $Y$ is called a *correspondence*, if the restrictions of the canonical projections $\pi_X$ and $\pi_Y$ onto $R$ are surjections. By $\mathcal{R}(X,Y)$ we denote the set of all correspondences between $X$ and $Y$.

Let $X$ and $Y$ be metric spaces, then for every relation $\sigma \in \mathcal{P}(X,Y)$ its *distortion* $\operatorname{dis}\sigma$ is defined as

$$\operatorname{dis}\sigma = \sup\Big\{\big||xx'| - |yy'|\big| : (x,y) \in \sigma,\ (x',y') \in \sigma\Big\}.$$

The following result is well-known.

**Proposition 2.1** *([2]). For any metric spaces $X$ and $Y$ we have*

$$d_{GH}(X,Y) = \frac{1}{2}\inf\big\{\operatorname{dis} R \mid R \in \mathcal{R}(X,Y)\big\}.$$