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Chain development of metric compacts ☆

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A B S T R A C T

Chain distance between points in a metric space is defined as the infimum of ε such that there is an ε-chain connecting these points. We call a mapping of a metric compact into the real line a chain development if it preserves chain distances. We give a criterion of existence of the chain development for metric compacts. We prove the diameter of any chain development of a given compact to be the same iff the compact is countable.

Notions and basic facts. Let (X, d) be a metric space. We call a sequence of points $x = x_0, x_1, x_2, \dots, x_n = y$ an ε-chain if $d(x_i, x_{i+1}) \leqslant \varepsilon$ for all i. Define *chain distance* $c(x, y)$ as the infimum of ε such that there exists an ε-chain from x to y.

Chain distance satisfies strong triangle inequality: $c(x, z) \leqslant \max(c(x, y), c(y, z))$; hence it is ultrametric if it does not degenerate. Obviously, $c = d$ if d is already ultrametric.

Definition. A function $f : X \to \mathbb{R}$ is called *chain development* if f preserves chain distance:

$$c(x, y) = \tilde{c}(f(x), f(y)) \quad \text{for } x, y \in X,$$

where c is the chain distance on (X, d) and $\tilde{c}$ is the chain distance on the set $f(X)$ with usual distance $\tilde{d}(s, t) = |s - t|$.

Chain development was firstly introduced by E.V. Schepin for finite sets as a tool for fast hierarchical cluster analysis. Note that chain development always exists for finite spaces and can be effectively constructed using minimum weight spanning tree of the corresponding graph; see [1] and [2, Section 4] for more details.

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An equivalent construction appeared in the paper [3] by A.F. Timan and I.A. Vestfid: they proved that points of any finite ultrametric space can be enumerated in a sequence $x_1, \dots, x_n$ such that $c(x_i, x_j) = \max(c(x_i, x_j), c(x_j, x_k))$ for $i < j < k$.

The goal of this paper is to discuss some properties of chain development for infinite spaces. So, there are compacts with no chain developments, e.g. the square $C \times C$ of a Cantor set. Necessary and sufficient condition of existence of chain developments is given below in Theorem 2.

By *diameter of a chain development* $f\colon X \to \mathbb{R}$ we mean $\operatorname{diam} f(X) = \sup f(X) - \inf f(X)$. It is proven in [1] that for finite spaces X the diameter of chain developments is determined uniquely. It turns out that this is not true in general case.

Theorem 1. *Let (X, d) be a compact metric space. Then the diameter of chain developments (if there are any) is determined uniquely if and only if X is countable.*

Throughout this paper by (Z, d) we denote a zero-dimensional compact metric space. We focus on such spaces because study of chain developments for arbitrary compacts essentially reduces to the zero-dimensional case.[1] We have the following property:

(i) (Z, c) is an ultrametric space, i.e. chain distance does not degenerate.

Indeed, take $x, y \in Z$. The set $\{x\}$ is a connected component, hence $x \in U \not\ni y$ for some closed open set U, so

$$c(x, y) \geqslant \min_{\substack{u \in U \\ v \in X \setminus U}} d(u, v) > 0.$$

The transition from metric d to ultrametric c (which can be seen as a functor) preserves topology:

(ii) The identity map $\mathrm{id}\colon Z \to Z$ is a homeomorphism between (Z, d) and (Z, c).

Indeed, id is 1-Lipshitz ($c(x, y) \leqslant d(x, y)$), hence it is a continuous bijection from compact to Hausdorff space, hence a homeomorphism.

(iii) Any chain development $f\colon Z \to \mathbb{R}$ is continuous (with usual topology on $\mathbb{R}$). Hence, $f(Z)$ is compact and f is a homeomorphism between Z and $f(Z)$.

Let $x_n \to x^*$ in Z; prove that $t_n := f(x_n) \to t^* =: f(x^*)$. Suppose that $t_n \not\to t^*$, say, $t_n > t^* + \varepsilon$ for some $\varepsilon > 0$. If there are no points of $f(Z)$ in $(t^*, t^* + \varepsilon)$, then $\tilde{c}(t_n, t^*) \geqslant \varepsilon$ (where $\tilde{c}$ is the chain distance on $f(Z)$). And if there is some $t = f(x) \in (t^*, t^* + \varepsilon)$, then $\tilde{c}(t_n, t^*) \geqslant \tilde{c}(t, t^*) = c(x, x^*) > 0$. In both cases $\tilde{c}(t_n, t^*) \not\to 0$, which contradicts that $\tilde{c}(t_n, t^*) = c(x_n, x^*) \leqslant d(x_n, x^*) \to 0$. So, f is continuous.

The chain distance on a compact $K \subset \mathbb{R}$ is determined by the lengths of the intervals of the open set $U_K := [\min K, \max K] \setminus K$.

(iv) Chain distance between points s, t of K is equal to the maximal length of the intervals of U_K, lying between s and t.

Existence of chain development. There is a well-known correspondence between ultrametric spaces and labeled trees; here we describe it for our purposes. Let (X, d) be a compact metric space; we will construct a

[1] One can identify points of (X, c) with $c(x, y) = 0$ to obtain zero-dimensional ultrametric compact (Z_X, c); a chain development of (X, d) exists if and only if there is a chain development of (Z_X, c).