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Uniform powers of compacta and the proximal game

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A B S T R A C T

The countable uniform power (or uniform box product) of a uniform space X is a special topology on ${}^{\omega}X$ that lies between the Tychonoff topology and the box topology. We solve an open problem posed by P. Nyikos showing that if X is a compact proximal space then the countable uniform power of X is also proximal (although it is not compact). By recent results of J.R. Bell and G. Gruenhage this implies that the countable uniform power of a Corson compactum is collectionwise normal, countably paracompact and Fréchet–Urysohn. We also give some results about first countability, realcompactness in countable uniform powers of compact spaces and explore questions by P. Nyikos about semi-proximal spaces.

1. Introduction

All spaces discussed in this paper will be assumed to be uniformizable (equivalently, Tychonoff, see [5, Theorem 8.1.20]).

Let $\langle X, \mathfrak{U}\rangle$ be a uniform space. For each $U \in \mathfrak{U}$, let

$$\widetilde{U} = \{\langle f, g\rangle : f, g \in {}^{\omega}X \text{ and } \forall n < \omega\ (\langle f(n), g(n)\rangle \in U)\}.$$

Let $\widetilde{\mathfrak{U}}$ be the uniformity on ${}^{\omega}X$ with base $\{\widetilde{U} : U \in \mathfrak{U}\}$. Let $\prod_u^{\omega}\langle X, \mathfrak{U}\rangle$ denote the topological space ${}^{\omega}X$ with the topology generated by $\widetilde{\mathfrak{U}}$. This space was introduced in [2] and called the countable uniform box

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product of X. We will call $\prod_u^\omega \langle X, \mathfrak{U}\rangle$ the countable uniform power of $\langle X, \mathfrak{U}\rangle$. If X has only one compatible uniformity, as is the case when X is compact [5, Theorem 8.3.12], we will only write $\prod_u^\omega X$.

Apparently the countable uniform power was discussed by Scott Williams during the 9th Prague International Topological Symposium (2001), where he asked whether the countable uniform power of a compact space is normal. We also remark that the uniformity $\widetilde{\mathfrak{U}}$ is also described in [5, p. 440] and called the uniformity of uniform convergence (although it is in fact defined for more general products).

In [2], Bell showed that if X is any Fort space (the one-point compactification of a discrete space), then $\prod_u^\omega X$ is collectionwise normal and countably paracompact. Later, in [3] she proved the corresponding results when X is the ω-power of a Fort space. In this last paper, Bell also defined the class of proximal spaces that encompasses both of these proofs. It turns out that all metric spaces are proximal and in fact proximality is preserved under subspaces, countable products and Σ-products (all proved in [3]). The main feature of proximal spaces is that they have some nice properties.

1.1. Theorem. *[3] If X is a proximal space, then X is collectionwise normal, countably paracompact and Fréchet–Urysohn.*

So in fact, in [2] and [3] it is just shown that if X is either a Fort space or the ω-power of a Fort space, then $\prod_u^\omega X$ is proximal. P. Nyikos has shown [8, Example 2.4] that there is a uniformity $\mathfrak{U}$ on $X = \omega \times (\omega+1)$ such that $\prod_u^\omega \langle X, \mathfrak{U}\rangle$ is not Fréchet-Urysohn. Thus, in the following we will restrict to compact spaces.

Notice that since X can be embedded as a closed subspace of $\prod_u^\omega X$, then proximality of $\prod_u^\omega X$ implies proximality of X. Problem 10.3 in [3] asks if it is possible to prove that $\prod_u^\omega X$ is proximal whenever X is proximal. In this paper we answer this Question in the affirmative (for the class of compacta).

1.2. Theorem. *If X is a proximal compactum, then $\prod_u^\omega X$ is proximal.*

After this, in section 4 we give a characterization of first countability of uniform powers.

In section 5 we give some observations on $\prod_u^\omega (\omega_1 + 1)$, we still don't know whether this space is normal (notice that $\omega_1 + 1$ is not proximal because it is not Fréchet–Urysohn). Thus, the general question still remains open.

1.3. Question. (S. Williams) Is $\prod_u^\omega X$ normal whenever X is compact?

In section 6 we consider semi-proximal spaces, a class of spaces defined by Nyikos in [8]. Being a weakening of proximality, it is natural to wonder which properties of proximal spaces are also held by semi-proximal spaces. In particular, Nyikos has asked whether semi-proximal spaces are normal [8, Problem 13]. We explore this question for products of subspaces of ω_1.

Finally, in section 7, we consider a variation of the proximal game, defined for topological spaces, which we call the *topological proximal game*. We compare the topological proximal game with the proximal game and show that they are not equivalent.

2. Preliminaries

One of the motivations for the definition of the uniform product is that its topology lies between the Tychonoff topology and the box topology. Many topological properties that are known to be preserved by Tychonoff products are completely lost for box products.

Consider ${}^{\omega}(\omega + 1)$. With the Tychonoff topology, this product is homeomorphic to the Cantor set so it is metrizable and compact. However, with the box topology this product is not even Fréchet and it is unknown if it is normal in ZFC. Thus, the uniform product presents an intermediate topology that has just begun to be studied.