Topology and its Applications 221 (2017) 449–464

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – The Mexican International Conference on Topology and Its Applications (MICTA-2014)

Whitney blocks

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ARTICLE INFO

Article history:
Received 19 August 2015
Received in revised form 12 November 2015
Accepted 26 November 2015
Available online 2 March 2017

MSC:
54B20
54F15

Keywords:
Absolute retract
Aposyndesis
Arc-smoothness
Arcwise connectedness
Continuum
Contractibility
Hyperspaces
Local connectedness
Mutual aposyndesis
Neighborhood retract
Unicoherence
Whitney blocks
Whitney levels
Whitney maps
Zero-dimensional closed set aposyndesis

ABSTRACT

Let $C(X)$ be the hyperspace of subcontinua of a continuum X. A Whitney block in $C(X)$ is a set of the form $\mu^{-1}([s,t])$, where $\mu : C(X) \to [0,1]$ is a Whitney map and $0 \leq s < t < 1$. In this paper we study topological properties P satisfying the implication: if X has property P, then each Whitney block in $C(X)$ has property P. We also consider the converse implication.
We study properties related to local connectedness, arcwise connectedness, property of Kelley, contractibility, aposyndesis, unicoherence, etc.

1. Introduction

A *continuum* is a compact connected metric space with more than one point. Given a continuum X, a *subcontinuum* of X is a closed, nonempty and connected subset of X. That is, $A \subset X$ is a subcontinuum of X if either A is a continuum with the induced topology or A is a one-point set. We consider the hyperspace

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http://dx.doi.org/10.1016/j.topol.2017.02.020
0166-8641/

$C(X)$ defined as

$$C(X) = \{A \subset X : A \text{ is a subcontinuum of } X\}.$$

The hyperspace $C(X)$ is considered with the Hausdorff metric H (see [6, Theorem 2.2]).

We also consider the hyperspace of singletons $F_1(X) = \{\{p\} \in C(X) : p \in X\}$, which is isometric to X.

A *Whitney map* for $C(X)$ is a continuous function $\mu : C(X) \to [0, 1]$ such that:

(a) $\mu(\{p\}) = 0$ for each $p \in X$,

(b) if $A, B \in C(X)$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$, and

(c) $\mu(X) = 1$.

It is known [6, Theorem 13.4] that for every continuum X, $C(X)$ admits Whitney maps.

A *Whitney level for* $C(X)$, is a set of the form $\mu^{-1}(t)$, where $0 < t < 1$.

A *Whitney block for* $C(X)$, is a set of the form $\mu^{-1}([s, t])$, where $0 \leq s < t \leq 1$.

A topological property P is *induced to Whitney blocks* provided that the following implication holds: if X has property P, then each Whitney block for $C(X)$ has property P.

Whitney levels have been widely studied. Many properties and references about Whitney levels can be found in the book [6]. In this paper and in [2] we introduce a systematic study on Whitney blocks.

Given a metric d for X and $A \subset X$, we denote by $\text{diam}_d(A)$, the diameter of A. If $\varepsilon > 0$, the set $C_{d,\varepsilon}(X) = \{A \in C(X)\text{: } \text{diam}_d(A) \leq \varepsilon\}$ is called *a small-point hyperspace* of X.

Hyperspaces of the form $C_{d,\varepsilon}(X)$ have been studied by E.L. McDowell and B.E. Wilder [8]; E.L. McDowell [9]; and M.E. Aguilera and A. Illanes [1]. For small ε, the hyperspace $C_{d,\varepsilon}(X)$ can be thought as an approximation of the space $F_1(X)$ (and then of X), so one can expect that some topological properties on $C_{d,\varepsilon}(X)$ are reflected on X and vice versa.

In [1,8,9], the following types of questions have been considered.

(1) What topological properties have all the hyperspaces of the form $C_{d,\varepsilon}(X)$?

(2) What topological properties on $C_{d,\varepsilon}(X)$, for small ε, can be deduced from supposing some topological properties on X?

(3) What topological properties on X can be deduced from supposing some topological properties on $C_{d,\varepsilon}(X)$, for small ε?

As it has been shown, some topological properties on $C_{d,\varepsilon}(X)$ strongly depend on the metric d. For example, in [9] it is shown a locally connected space X for which there are non-local connected hyperspaces $C_{d,\varepsilon}(X)$ for an appropriate metric d and for some ε.

Looking for better approximations of $F_1(X)$ in $C(X)$, in this paper we study the hyperspaces of the form $\mu^{-1}([s, t])$, where μ is a Whitney map for $C(X)$ and $0 \leq s < t \leq 1$. In the case that $s = 0$, they are *initial Whitney blocks*.

We consider questions (1), (2) and (3) for Whitney blocks instead of small-point hyperspaces, and we show that, for this case, better answers can be obtained.

In this paper we consider properties like, local connectedness, arcwise connectedness, Kelley property, contractibility, aposyndesis, unicoherence, etc.

Finally, we mention that a number of authors have studied the corresponding questions (1), (2) and (3) for *Whitney levels*, the interested reader can found a very complete information on this topic in [6, Chapter VIII].

2. General properties

For a continuum X and $A, B \in C(X)$ such that $A \subset B$, an *order arc from A to B* is a continuous function $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and $\alpha(u) \subset \alpha(v)$ if $0 \leq u \leq v \leq 1$. It is known [6, Theorem 14.6] that if $A, B \in C(X)$ and $A \subset B$, then there exist order arcs from A to B.