Topology and its Applications 221 (2017) 491–500

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – The Mexican International Conference on Topology and Its Applications (MICTA-2014)

Completion of pre-quasi-uniform spaces

Adalberto García-Máynez [a], Adolfo Pimienta Acosta [b,*,1]

[a] *Instituto de Matemáticas, Universidad Nacional Autónoma de México, Area de la Investigación Científica Circuito Exterior, Ciudad Universitaria Coyoacán, 04510, México, D.F., Mexico*
[b] *Departamento de Matemáticas, Universidad Autónoma Metropolitana, prolongación canal de miramontes #3855, Col. Ex-Hacienda San Juan de Dios, Delegación Tlalpan, C.P. 14387, México, D.F., Mexico*

A R T I C L E   I N F O

Article history:
Received 25 November 2014
Received in revised form 2 March 2015
Accepted 16 March 2015
Available online 10 February 2017

MSC:
primary 54A20, 54D30, 54D35, 54D70, 54D80, 54E15

Keywords:
Connector
Unimorphism
Quasi-uniformity
Pre-uniformity
Pre-quasi-uniformity
Concrete
U-selective map
Uniformity
Filter
$\mathcal{U}$-Cauchy filter
Minimal $\mathcal{U}$-Cauchy filter
Mixing filters
Weakly round filters
Round filter

A B S T R A C T

The purpose in this paper is to prove the existence of a canonical-pre-quasi-uniform extension $(X,\mathcal{U})\hat{}$ of a pre-quasi-uniform space $(X,\mathcal{U})$. Most of the concepts used in quasi-uniform spaces may be applied to pre-quasi-uniform spaces. These spaces were introduced by the first author and developed by the second in his doctoral thesis. We shall study the main properties of this class of spaces which contains the class of quasi-uniform spaces.

1. Preliminaries

For terminology and notation we follow [1,2,5,6] (except when mentioning the contrary).

* Corresponding author.
E-mail addresses: agmaynez@matem.unam.mx (A. García-Máynez), pimienta@xanum.uam.mx (A. Pimienta Acosta).

[1] The second author was supported in part by Universidad de la Costa (CUC – www.cuc.edu.co) under grant of the Department of Exact and Natural Sciences, address street 58 # 55 - 66, Barranquilla, Colombia.

http://dx.doi.org/10.1016/j.topol.2017.02.025
0166-8641/

In order to make this article accessible to the nonspecialist we begin by recalling some basic concepts of the theory of quasi-uniform spaces.

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ such that:

(i) For each $U \in \mathcal{U}$, $\Delta(X) = \{(x,x): x \in X\} \subseteq U$.
(ii) For each $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V \circ V = V^2 \subseteq U$.
(Here $V \circ V = \{(x,y) \in X \times X$: There exists $z \in X$ such that $(x,z) \in V$ and $(z,y) \in V\}$.)

A (sub)base $\mathcal{B}$ for $\mathcal{U}$ is *transitive* provided that for each $B \in \mathcal{B}$, $B \circ B = B$. A quasi-uniformity with a transitive base is called a *transitive quasi-uniformity*.

The pair $(X,\mathcal{U})$ is called a *quasi-uniform space*. Let us note that for any quasi-uniformity $\mathcal{U}$ on X, the filter $\mathcal{U}^{-1} = \{U^{-1}: U \in \mathcal{U}\}$ is also a quasi-uniformity on X called the conjugate of $\mathcal{U}$. (Of course, $U^{-1} = \{(x,y) \in X \times X: (y,x) \in U\}$.)

Furthermore $\mathcal{U}^* = \mathcal{U} \vee \mathcal{U}^{-1}$ is a *uniformity* on X.

A quasi-uniformity $\mathcal{U}$ on a set X is called *totally bounded* provided that for each $U \in \mathcal{U}$ there exists a finite cover $\mathcal{A}$ of X such that $A \times A \subseteq U$ whenever $A \in \mathcal{A}$ (equivalently, if the uniformity $\mathcal{U}^*$ is *precompact*, i.e. for each $U \in \mathcal{U}^*$ there is a finite subset F of X such that $U(F) = X$).

Let X be a set. A set $U \subseteq X \times X$ is a *connector* of X if $\Delta(X) = \{(x,x): x \in X\} \subseteq U$ or, equivalently, if U is a reflexive relation on X.

Definition 1.1. Let X be a non-empty set and let $\mathcal{U}$ be a filter in $X \times X$ whose elements are connectors of X. The topology $\tau_{\mathcal{U}}$ induced by $\mathcal{U}$ is:

$$\tau_{\mathcal{U}} = \{V \subseteq X: \forall x \in V, \exists U \in \mathcal{U} \text{ such that } U(x) \subset V\}.$$

Here $U(x) = \{y \in X: (x,y) \in U\}$, when $U \in \mathcal{U}$ and $x \in X$.

Definition 1.2. Let X be a non-empty set. A filter $\mathcal{U}$ on $X \times X$ consisting of connectors of X is a *pre-quasi-uniformity* on X if the following condition is fulfilled:

$$\forall U \in \mathcal{U},\ \exists V \in \mathcal{U} \text{ such that } V(x) \subseteq intU(x)\ \forall x \in X. \tag{1}$$

A pre-quasi-uniform space is a pair $(X,\mathcal{U})$, where X is a non-empty set and $\mathcal{U}$ is a pre-quasi-uniformity on X.

It is clear that every quasi-uniformity on X is a pre-quasi-uniformity on X.

Definition 1.3. An *indexed cover* of X is a map $\varphi: X \to \mathcal{P}(X)$ where $x \in \varphi(x)$ for each $x \in X$. A more familiar notation is $\{U_x: x \in X\}$, where $x \in U_x = \varphi(x)$ for each $x \in X$.

Example 1.4. Let ξ be an arbitrary cover of X. The *barycentric cover* ξ^Δ is the indexed cover $\xi^\Delta = \{St(x,\xi): x \in X\}$, where $St(x,\xi) = \cup\{G \in \xi: x \in G\}$. In this case:

$$E(\xi^\Delta) = \cup\{G \times G: G \in \xi\}$$

is a symmetric connector.

The cobarycentric cover ξ^∇ is the index cover

$$\xi^\nabla = \{Cost(x,\xi): x \in X\},$$