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Inclusions of characterized subgroups

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ABSTRACT

A subgroup H of $\mathbb{R}$ is characterized if $H = \tau_{\mathbf{u}}(\mathbb{R}) := \{x \in \mathbb{R} : u_n x \to 0 \bmod \mathbb{Z}\}$ for some sequence $\mathbf{u}$ in $\mathbb{R}$. Given two sequences $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}$, we find conditions under which $\tau_{\mathbf{u}}(\mathbb{R})$ is contained or not in $\tau_{\mathbf{v}}(\mathbb{R})$. As a by-product of our main theorems, we find a known result by Eliaš on inclusions of characterized subgroups of $\mathbb{T}$, motivated by problems in harmonic analysis.

1. Introduction

Let G be a topological abelian group and denote by $\widehat{G}$ the group of all continuous characters $\chi : G \to \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is endowed with the compact quotient topology inherited from $\mathbb{R}$. Following [19], for a sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ in $\widehat{G}$, let

$$s_{\mathbf{u}}(G) := \{x \in G : u_n(x) \to 0\}.$$

A subgroup H of G is called *characterized* if $H = s_{\mathbf{u}}(G)$ for some sequence $\mathbf{u}$ in $\widehat{G}$.

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The characterized subgroups were considered almost exclusively for metrizable compact abelian groups (e.g., see [5,18]); only recently, the case of general compact abelian groups was given full attention in [13], and the case of abelian topological groups in [16] (see also [24,25]).

The fundamental and starting case remains when $G = \mathbb{T}$ (e.g., see [6,27,29]); the characterized subgroups of $\mathbb{T}$ were studied also in relation to Diophantine approximation, dynamical systems and ergodic theory (see [6,30,35] and the survey [24]).

Moreover, it is worth pointing out that, since $\widehat{\mathbb{T}}$ can be identified with $\mathbb{Z}$, we may assume that a sequence $\mathbf{u}$ in $\widehat{\mathbb{T}}$ is a sequence in $\mathbb{Z}$. Then $s_{\mathbf{u}}(\mathbb{T})$ coincides with the subgroup

$$t_{\mathbf{u}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \to 0\}$$

of all *topologically* $\mathbf{u}$-*torsion elements* of $\mathbb{T}$. The concept of topologically $\mathbf{u}$-torsion element generalizes that of topologically torsion element (for $u_n = n!$) and that of topologically p-torsion element (for $u_n = p^n$), which were introduced to study the structure of topological groups and in particular of locally compact abelian groups (see [2,8,20,33] and the survey [11]). A complete description of the subgroups $t_{\mathbf{u}}(\mathbb{T})$ was found in [17,12] for sequences $\mathbf{u}$ in $\mathbb{N}$ such that u_n divides u_{n+1} for every $n \in \mathbb{N}$.

We consider here the characterized subgroups of $\mathbb{R}$. Since $\widehat{\mathbb{R}}$ is topologically isomorphic to $\mathbb{R}$, we can identify a sequence $\mathbf{u}$ in $\widehat{\mathbb{R}}$ with a sequence in $\mathbb{R}$; then $s_{\mathbf{u}}(\mathbb{R})$ coincides with the subgroup

$$\tau_{\mathbf{u}}(\mathbb{R}) := \{x \in \mathbb{R} : u_n x \to 0 \mod \mathbb{Z}\}.$$

We would like to underline that, if $\mathbf{u}$ is a sequence in $\mathbb{Z}$, the examination of $\tau_{\mathbf{u}}(\mathbb{R})$ includes that of $t_{\mathbf{u}}(\mathbb{T})$; in fact, in this case $\tau_{\mathbf{u}}(\mathbb{R}) = \pi^{-1}(t_{\mathbf{u}}(\mathbb{T}))$, where $\pi : \mathbb{R} \to \mathbb{T}$ is the canonical projection.

Characterized subgroups of $\mathbb{R}$ were studied in relation to uniform distribution of sequences modulo $\mathbb{Z}$ by Kuipers and Niederreiter in the book [28], where [28, Theorem 7.8] shows that $\tau_{\mathbf{u}}(\mathbb{R})$ has Lebesgue measure zero if $\mathbf{u}$ is a sequence in $\mathbb{R}$ not converging to 0 in $\mathbb{R}$ (they give credit to Schoenberg for this result, see [32]). Moreover, Borel proved in [7, Proposition 2] that if $H = \tau_{\mathbf{v}}(\mathbb{R})$ is a non-trivial proper characterized subgroup of $\mathbb{R}$, then there exist $\gamma \in \mathbb{R}$ and a strictly increasing sequence $\mathbf{u}$ in $\mathbb{N}$ such that $\gamma H = \tau_{\mathbf{u}}(\mathbb{R})$. This underlines the strict relation between characterized subgroups of $\mathbb{R}$ and characterized subgroups of $\mathbb{T}$. Borel proved also that every countable subgroup of $\mathbb{R}$ is characterized and left open the general question of a complete description of the characterized subgroups of $\mathbb{R}$.

In this paper, under some restrictions on the sequences, we find conditions ensuring the inclusion of one characterized subgroup of $\mathbb{R}$ into another characterized subgroup of $\mathbb{R}$. In particular, we consider characterized subgroups $\tau_{\mathbf{u}}(\mathbb{R})$ of $\mathbb{R}$ always under the assumption that $\mathbf{u}$ is in $\mathbb{R} \setminus \{0\}$ and $|q_n^{\mathbf{u}}| \to +\infty$, where

$$q_n^{\mathbf{u}} := \tfrac{u_n}{u_{n-1}} \ (n \in \mathbb{N}) \text{ and } u_0 = 1.$$

Thus, the cardinality of $\tau_{\mathbf{u}}(\mathbb{R})$ is $\mathfrak{c}$ (see Remark 2.1). Moreover, we can always assume that such sequences are in $\mathbb{R}_+$, since for any sequence $\mathbf{w}$ in $\mathbb{R}$ we have $\tau_{\mathbf{w}}(\mathbb{R}) = \tau_{|\mathbf{w}|}(\mathbb{R})$ where $|\mathbf{w}| := (|w_n|)_{n\in\mathbb{N}}$.

The problem of reciprocal inclusions of characterized subgroups has the following topological motivation, described in more detail in Section 2. Recall that a topological abelian group G is *totally bounded* if for every non-empty open set U in G there exists a finite subset F of G such that $G = U + F$; moreover, G is *precompact* if it is Hausdorff and totally bounded. For two sequences $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}$ such that $|q_n^{\mathbf{u}}| \to +\infty$, we see in Corollary 2.7 that the condition $\tau_{\mathbf{u}}(\mathbb{R}) \not\subseteq \tau_{\mathbf{v}}(\mathbb{R})$ is equivalent to the existence of a non-metrizable precompact group topology $\mathcal{T}$ on $\mathbb{R}$ such that $u_n \to 0\ (\mathcal{T})$ and $v_n \not\to 0\ (\mathcal{T})$ (i.e., $u_n \to 0$ in $(\mathbb{R}, \mathcal{T})$ and $v_n \not\to 0$ in $(\mathbb{R}, \mathcal{T})$), and also to the fact that $v_n \not\to 0\ (\sigma_{\mathbf{u}})$ where $\sigma_{\mathbf{u}}$ is the finest precompact group topology such that $u_n \to 0\ (\sigma_{\mathbf{u}})$ (see (2.1) for the definition of $\sigma_{\mathbf{u}}$). The topology $\sigma_{\mathbf{u}}$ considered in [4,19] was investigated in [14], where also ss-precompact groups were studied (a precompact group topology $\mathcal{T}$ on an abelian group G is *ss-precompact* if there exists a sequence $\mathbf{u}$ in G such that $\mathcal{T} = \sigma_{\mathbf{u}}$).