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Separation and cardinality – Some new results and old questions

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It is well known that separation axioms together with some local and global cardinal invariants lead to restriction of the cardinality of a given topological space. For an extensive survey, one can look at [16]. We shall mention here just a few such results that are directly related to this paper. At the end we will discuss some long-standing open problems in cardinal invariant theory and give a possible way of approaching a solution.

1. Introduction and definitions

We shall follow notations from [18] and [12]. We use $\mathbb{N}^+$ for the set of positive integers. In 1969 Arhangel'skii proved his famous result, answering a 50 years old question posed by P.S. Alexandroff:

1.1 Theorem *([2]). If X is Hausdorff then $|X| \leqslant 2^{L(X)\chi(X)}$.*

And posed two famous questions:

1 Question. Is the cardinality of T_1 first countable Lindelöf spaces less than or equal continuum?

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2 Question. Is the cardinality of Hausdorff, Lindelöf spaces with countable pseudocharacter less than or equal continuum?

Both questions have been extensively studied and for the second one counterexamples under certain set-theoretic assumptions have been found. For the compact case we have a positive answer for the first question, given by A. Gryzlov [14] even when the pseudocharacter is countable. But in general we have no progress in Question 1 so far. In this paper we shall discuss a possible approach in finding a solution. Both questions gave rise to a number of results some of which generalize Arhangel'skii's theorem. We shall list some of them that are related to the present paper.

Let us recollect some definitions and results for completeness:

1.2 Definition. For a topological space X and $Y \subseteq X$, let:

$$aL(Y,X) := \omega + \min\{\tau : \text{ for each open in } X \text{ cover } \gamma \text{ of } Y\ \exists\gamma' \in [\gamma]^{\leqslant\tau} \text{ such that } Y \subseteq \bigcup\{\overline{U} : U \in \gamma'\}\},$$
$$aL_c(X) := \omega + \sup\{aL(Y,X) : Y \text{ is closed in } X\},$$
$$aL(X) := aL(X,X) \qquad \text{(Almost Lindelöf number)}$$

1.3 Definition. For a topological space X and $Y \subseteq X$, let:

$$wL(Y,X) := \omega + \min\{\tau : \text{ for each open in } X \text{ cover } \gamma \text{ of } Y\ \exists\gamma' \in [\gamma]^{\leqslant\tau} \text{ such that } Y \subseteq \overline{\bigcup\{U : U \in \gamma'\}}\},$$
$$wL_c(X) := \omega + \sup\{wL(Y,X) : Y \text{ is closed in } X\},$$
$$wL(X) := wL(X,X) \qquad \text{(weak Lindelöf number).}$$

We list (almost chronologically) the following results, related to the questions discussed here:

1.4 Theorem *([27]). If X is Hausdorff then $|X| \leqslant 2^{aL_c(X)\chi(X)}$.*

1.5 Theorem *([27]). If X is regular then $|X| \leqslant 2^{aL(X)\chi(X)}$.*

The above theorem was improved in 1988 by A. Bella and F. Cammaroto:

1.6 Theorem *([5]). If X is Urysohn then $|X| \leqslant 2^{aL(X)\chi(X)}$.*

Using the weaker form of Lindelöfness but a stronger separation axiom, Bell, Gingsburgh and Woods proved:

1.7 Theorem *([6]). If X is normal then $|X| \leqslant 2^{wL(X)\chi(X)}$.*

They asked if their result can be improved for regular spaces and this question is still open. In that direction Arhangel'skii proved:

1.8 Theorem *([3]). If X is regular then $|X| \leqslant 2^{wL_c(X)\chi(X)}$.*

He asked if this result is true for Hausdorff spaces. This question is still open as well, but in 1993, O. Alas improved Arhangel'skii's result to Urysohn spaces:

1.9 Theorem *([1]). If X is Urysohn then $|X| \leqslant 2^{wL_c(X)\chi(X)}$.*