Topology and its Applications 218 (2017) 1–18

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On Riemann surfaces of genus g with $4g$ automorphisms [☆]

Emilio Bujalance [a], Antonio F. Costa [a,*], Milagros Izquierdo [b]

[a] *Departamento de Matemáticas Fundamentales, UNED, Paseo Senda del Rey 9, 28040 Madrid, Spain*
[b] *Matematiska institutionen, Linköpings universitet, 581 83 Linköping, Sweden*

A R T I C L E   I N F O

Article history:
Received 7 April 2016
Accepted 19 December 2016
Available online 23 December 2016

MSC:
primary 30F10
secondary 14H15, 30F60

Keywords:
Riemann surfaces
Automorphisms
Moduli spaces

A B S T R A C T

We determine, for all genus $g \geq 2$ the Riemann surfaces of genus g with exactly $4g$ automorphisms. For $g \neq 3, 6, 12, 15$ or 30, these surfaces form a real Riemann surface $\mathcal{F}_g$ in the moduli space $\mathcal{M}_g$: the Riemann sphere with three punctures. We obtain the automorphism groups and extended automorphism groups of the surfaces in the family. Furthermore we determine the topological types of the real forms of real Riemann surfaces in $\mathcal{F}_g$. The set of real Riemann surfaces in $\mathcal{F}_g$ consists of three intervals its closure in the Deligne–Mumford compactification of $\mathcal{M}_g$ is a closed Jordan curve. We describe the nodal surfaces that are limits of real Riemann surfaces in $\mathcal{F}_g$.

1. Introduction

Given a linear expression like $ag + b$, where a, b are fixed integers, it is very difficult to claim precise information on the (compact) Riemann surfaces of genus $g \geq 2$ with automorphism groups of order $ag + b$: i.e. are there Riemann surfaces in these conditions?, how many?, which are their automorphism groups? For instance, there are many works about Hurwitz surfaces, i.e. surfaces of genus g with group of automorphisms of order $84g - 84$ (maximal order), but there is no complete answer to the above questions. Surprisingly we shall give an almost complete answer (up to a finite number of genera g) to all questions on Riemann surfaces of genus g with $4g$ automorphisms.

For each integer $g \geq 2$ we find an equisymmetric (complex)-uniparametric family $\mathcal{F}_g$ of Riemann surfaces of genus g having (full) automorphism group of order $4g$. The maximal order $ag + b$ of the automorphism group of equisymmetric and uniparametric families of Riemann surfaces appearing in all genera is $4g + 4$ and the second possible largest order is precisely $4g$ (this is a consequence of Riemann–Hurwitz formula).

☆ Acknowledgements: All authors are partially supported by the project MTM2014-55812-P.
* Corresponding author.
E-mail addresses: eb@mat.uned.es (E. Bujalance), acosta@mat.uned.es (A.F. Costa), miizq@mai.liu.se (M. Izquierdo).

http://dx.doi.org/10.1016/j.topol.2016.12.013
0166-8641/

If $g \neq 3, 6, 15$ all surfaces with $4g$ automorphisms are in the family $\mathcal{F}_g$ with one or two more exceptional surfaces in a few genera: $g = 3, 6, 12, 30$. For genera $g = 3, 6$ and 15 appear another exceptional uniparametric family. Finally for genera 3, 6, 12 and 30 there are one or two exceptional surfaces with $4g$ automorphisms.

The automorphism group of the surfaces in $\mathcal{F}_g$ is D_{2g} and the quotient $X/\mathrm{Aut}(X)$ is the Riemann sphere $\widehat{\mathbb{C}}$, the meromorphic function $X \to X/\mathrm{Aut}(X) = \widehat{\mathbb{C}}$ have four singular values of orders $2, 2, 2, 2g$.

Kulkarni [17] showed that, for any genus $g \equiv 0, 1, 2 \bmod 4$, there is a unique surface of genus g with full automorphism group of order $8(g+1)$ (the family of Accola–Maclachan [1,20]), and for $g \equiv -1 \bmod 4$, there is just another surface of genus g (the Kulkarni surface [17]). In [18] Kulkarni shows that, if $g \neq 3$ there is a unique Riemann surface of genus g admitting an automorphism of order $4g$, while for $g = 3$ there are two such surfaces (see also [9,16]). The surfaces in this last family have exactly $8g$ automorphisms, except for $g = 2$, where the surface has 48 automorphisms. For cyclic groups there are some cases where the order of the group determines the Riemann surface (see [18,21,15]). Analogous results are known for Klein surfaces: [4,7,8,3].

The family $\mathcal{F}_g$ contains surfaces admitting anticonformal automorphisms, forming the subset $\mathbb{R}\mathcal{F}_g$. These points in the moduli space correspond to Riemann surfaces given by the complexification of real algebraic curves. The extended groups of automorphisms of the surfaces in $\mathbb{R}\mathcal{F}_g$ (including the anticonformal automorphisms) are isomorphic either to $D_{2g} \times C_2$ or D_{4g}, and such groups contain anticonformal involutions, so the surfaces in $\mathbb{R}\mathcal{F}_g$ are real Riemann surfaces. The topological types of conjugacy classes of anticonformal involutions (real forms) of the real Riemann surfaces in $\mathcal{F}_g$ are either $\{+2, 0, -2, -2\}$, $\{-1, -1, -g, -g\}$, $\{0, 0, -2, -2\}$ if g is odd or $\{+1, 0, -1, -3\}$, $\{-1, -1, -g, -g\}$, $\{-2\}$ if g is even.

The family $\mathcal{F}_g$ is the Riemann sphere with three punctures, having an anticonformal involution whose fixed point set consists of three arcs a_1, a_2, b. Each one of these arcs is formed by the real Riemann surfaces in $\mathbb{R}\mathcal{F}_g$ with a different set of topological types of real forms. Adding three points to the surface $\mathcal{F}_g$ (one of them corresponds to the Wiman surface of type II, the other two are surfaces in $\widehat{\mathcal{M}}_g \setminus \mathcal{M}_g$) we obtain a compact Riemann surface $\overline{\mathcal{F}_g} \subset \widehat{\mathcal{M}}_g$, where $\overline{a_1 \cup a_2 \cup b}$ (the closure of $a_1 \cup a_2 \cup b$ in $\widehat{\mathcal{M}}_g$) is a closed Jordan curve. The space $\widehat{\mathcal{M}}_g$ is the Deligne–Mumford compactification of $\mathcal{M}_g$. As a consequence we have that $\overline{\mathbb{R}\mathcal{F}_g} \cap \mathcal{M}_g$ has two connected components.

2. Preliminaries

2.1. Non-Euclidean crystallographic groups

A *non-Euclidean crystallographic group* (or *NEC group*) Γ is a discrete group of isometries of the hyperbolic plane $\mathbb{D}$. We shall assume that an NEC group has a compact orbit space. If Γ is such a group then its algebraic structure is determined by its signature

$$(h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}). \tag{1}$$

The orbit space $\mathbb{D}/\Gamma$ is a surface, possibly with boundary. The number h is called the *genus* of Γ and equals the topological genus of $\mathbb{D}/\Gamma$, while k is the number of the boundary components of $\mathbb{D}/\Gamma$, and the sign is $+$ or $-$ according to whether the surface is orientable or not. The integers $m_i \geq 2$, called the *proper periods*, are the branch indices over interior points of $\mathbb{D}/\Gamma$ in the natural projection $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$. The bracketed expressions $(n_{i1}, \ldots, n_{is_i})$, some or all of which may be empty (with $s_i = 0$), are called the *period cycles* and represent the branchings over the ith boundary component of the surface. Finally the numbers $n_{ij} \geq 2$ are the *link periods*.

Associated with each signature there exists a *canonical presentation* for the group Γ. If the signature (1) has sign $+$ then Γ has the following generators:

$x_1, \ldots, x_r$ (elliptic elements),