Topology and its Applications 218 (2017) 66–92

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Some spaces of polynomial knots

Hitesh Raundal, Rama Mishra *

Department of Mathematics, Indian Institute of Science Education and Research, Dr. Homi Bhabha Road, Pashan, Pune 411008, India

ARTICLE INFO

Article history:
Received 12 April 2016
Received in revised form 5 October 2016
Accepted 29 December 2016
Available online 9 January 2017

MSC:
14P10
55P15
57M25
57R40
57R52

Keywords:
Polynomial knot
Polynomial representation
Homotopy
Isotopy

ABSTRACT

In this paper we study the topology of three different kinds of spaces associated to polynomial knots of degree at most d, for $d \geq 2$. We denote these spaces by $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$. For $d \geq 3$, we show that the spaces $\mathcal{O}_d$ and $\mathcal{P}_d$ are path connected and the space $\mathcal{O}_d$ has the same homotopy type as S^2. Considering the space $\mathcal{P} = \bigcup_{d\geq 2} \mathcal{O}_d$ of all polynomial knots with the inductive limit topology, we prove that it too has the same homotopy type as S^2. We also show that if two polynomial knots are path equivalent in $\mathcal{Q}_d$, then they are topologically equivalent. Furthermore, the number of path components in $\mathcal{Q}_d$ are in multiples of eight.

1. Introduction

Parameterizing knots has been useful in estimating some important knot invariants such as bridge index [4], superbridge index [12] and geometric degree [9]. Some of the interesting parameterizations are Fourier knots [10], polygonal knots [6] and polynomial knots [19]. The first two of them provide classical knots, whereas the polynomial parametrization gives long knots. In each parametrization there is a positive integer d associated to it. For Fourier knots and polynomial knots it is its *degree* and for polygonal knots it is its *edge number*. For each parametrization, one can study the space of all parametrized knots for a fixed d. Once we fix d, there will be only finitely many knots that can be parametrized with this d. However, in the space of parameterizations, one could study the topology and try to see if two parametrized knots in this space belong to the same path components or not. In [6], Calvo studied the spaces of polygonal knots.

* Corresponding author.
E-mail addresses: hiteshrndl@gmail.com (H. Raundal), ramamishra64@gmail.com (R. Mishra).

http://dx.doi.org/10.1016/j.topol.2016.12.020

We aim to study the topology of some spaces of polynomial knots. In [19], Vassiliev defined the space $\mathcal{V}_d$ to be the interior of the set of all smooth embeddings in the space $\mathcal{W}_d$ of all polynomial maps of the type $t \mapsto \left(t^d + a_{d-1}t^{d-1} + \cdots + a_1t,\ t^d + b_{d-1}t^{d-1} + \cdots + b_1t,\ t^d + c_{d-1}t^{d-1} + \cdots + c_1t\right)$. He pointed out that the space of long knots can be approximated by the spaces $\mathcal{V}_d$ for $d \geq 1$ (see [18]). Also, it was noted that if two polynomial knots belong to the same path component of $\mathcal{V}_d$ then they represent the same knot-type. It is felt that the converse of this may not be true. However, no counter example is known. Regarding the topology of the space $\mathcal{V}_d$, he showed that the space $\mathcal{V}_3$ is contractible and the space $\mathcal{V}_4$ is homology equivalent to S^1.

Later, Durfee and O'Shea [1] introduced a different space $\mathcal{K}_d$ which is the space of all knots $t \mapsto \left(a_0 + a_1t + \cdots + a_dt^d,\ b_0 + b_1t + \cdots + b_dt^d,\ c_0 + c_1t + \cdots + c_dt^d\right)$ such that $|a_d| + |b_d| + |c_d| \neq 0$. If two polynomial knots are path equivalent in $\mathcal{K}_d$, then they are topologically equivalent. Thus, the space $\mathcal{K}_d$ will have at least as many path components as knot-types that can be represented in it. In [13], a group of undergraduate students proved that $\mathcal{K}_5$ has at least 3 path components corresponding to the unknot, the right hand trefoil and the left hand trefoil respectively. Beyond this the topology of these spaces is not understood. Also, it is not clear whether two topologically equivalent knots in this space necessarily belong to the same path component or not. Composing a polynomial knot with a simple polynomial automorphism of $\mathbb{R}^3$ can reduce the degree of two of the components and the resulting polynomial knot will be topologically equivalent to the earlier one. Keeping this view in mind, we note that any polynomial knot given by $t \mapsto \left(a_0 + a_1t + \cdots + a_dt^d,\ b_0 + b_1t + \cdots + b_dt^d,\ c_0 + c_1t + \cdots + c_dt^d\right)$, for $d \geq 2$, is topologically equivalent to a polynomial knot $t \mapsto \big(f(t),\ g(t),\ h(t)\big)$ with $\deg(f) \leq d-2$, $\deg(g) \leq d-1$ and $\deg(h) \leq d$. This motivated us to study the topology of three interesting spaces namely: (1) the space $\mathcal{O}_d$ of all polynomial knots $t \mapsto \big(f(t),\ g(t),\ h(t)\big)$ with $\deg(f) \leq d-2$, $\deg(g) \leq d-1$ and $\deg(h) \leq d$, (2) the space $\mathcal{P}_d$ of all polynomial knots $t \mapsto \big(u(t),\ v(t),\ w(t)\big)$ with $\deg(u) < \deg(v) < deg(w) \leq d$, and (3) the space $\mathcal{Q}_d$ of all polynomial knots $t \mapsto \big(x(t),\ y(t),\ z(t)\big)$ with $\deg(x) = d-2$, $\deg(y) = d-1$ and $\deg(z) = d$. Using the theory of real semialgebraic sets (see [11] and [14]), we ensure that these spaces must have finitely many path components. We show that the space $\mathcal{O}_d$, for $d \geq 3$, has the same homotopy type as S^2. We also prove that the space $\mathcal{P}_d$, for $d \geq 3$, is path connected, whereas the space $\mathcal{Q}_d$ is not path connected. For the space $\mathcal{Q}_d$, for $d \geq 2$, we show that if two polynomial knots lie in the same path component then they are topologically equivalent. We provide a counter example that the converse of this is not true. Furthermore, we show that the space $\mathcal{P}$ of all polynomial knots, with the inductive limit topology coming from the stratification $\mathcal{P} = \bigcup_{d \geq 2} \mathcal{O}_d$, also has the same homotopy type as S^2.

This paper is organized as follows: Section 2 is about definitions and known results. We divide it in three subsections. In 2.1 and 2.2, we provide the basic terminologies and some known results related to knots and in particular polynomial knots which will be required in this paper. In 2.3, we discuss real semialgebraic sets and mention some important results from real semialgebraic geometry. In Section 3, we introduce our spaces $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ for $d \geq 2$ and check their basic topological properties. At the end of Section 3, we show that these spaces are homeomorphic to some semialgebraic subsets of $\mathbb{R}^{3d}$ and hence they have only finitely many path components. In Section 4, we prove our main results which are the following:

Theorem 4.5: *The space $\mathcal{P}_d$, for $d \geq 3$, is path connected.*

Theorem 4.6: *If two polynomial knots are path equivalent in $\mathcal{Q}_d$, then they are topologically equivalent.*

Theorem 4.10: *Every polynomial knot is isotopic to some trivial polynomial knot by a smooth isotopy of polynomial knots.*

Corollary 4.11: *Every polynomial knot is connected to a trivial polynomial knot by a smooth path in the space $\mathcal{P}$ of all polynomial knots.*

Theorem 4.13: *The space $\mathcal{O}_d$, for $d \geq 3$, has the same homotopy type as S^2.*

Corollary 4.15: *The space $\mathcal{P}$ of all polynomial knots has the same homotopy type as S^2.*