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Topological coarse shape homotopy groups

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ABSTRACT

Cuchillo-Ibanez et al. introduced a topology on the sets of shape morphisms between arbitrary topological spaces in 1999. In this paper, applying a similar idea, we introduce a topology on the set of coarse shape morphisms $Sh^*(X,Y)$, for arbitrary topological spaces X and Y. In particular, we can consider a topology on the coarse shape homotopy group of a topological space (X,x), $Sh^*((S^k,*),(X,x)) = \check{\pi}_k^*(X,x)$, which makes it a Hausdorff topological group. Moreover, we study some properties of these topological coarse shape homotopy groups such as second countability, movability and in particular, we prove that $\check{\pi}_k^{*^{top}}$ preserves finite product of compact Hausdorff spaces. Also, we show that for a pointed topological space (X,x), $\check{\pi}_k^{top}(X,x)$ can be embedded in $\check{\pi}_k^{*^{top}}(X,x)$.

1. Introduction and motivation

Suppose that (X,x) is a pointed topological space. We know that $\pi_k(X,x)$ has a quotient topology induced by the natural map $q : \Omega^k(X,x) \to \pi_k(X,x)$, where $\Omega^k(X,x)$ is the kth loop space of (X,x) with the compact-open topology. With this topology, $\pi_k(X,x)$ is a quasitopological group, denoted by $\pi_k^{qtop}(X,x)$ and for some spaces it becomes a topological group (see [5–7,15]).

Calcut and McCarthy [8] proved that for a path connected and locally path connected space X, $\pi_1^{qtop}(X)$ is a discrete topological group if and only if X is semilocally 1-connected (see also [6]). Pakdaman et al. [24] showed that for a locally $(n-1)$-connected space X, $\pi_n^{qtop}(X,x)$ is discrete if and only if X is

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semilocally n-connected at x (see also [15]). Fabel [12,13] and Brazas [6] presented some spaces for which their quasitopological homotopy groups are not topological groups. Moreover, despite Fabel's result [12] that says the quasitopological fundamental group of the Hawaiian earring is not a topological group, Ghane et al. [16] proved that the topological nth homotopy group of an n-Hawaiian like space is a prodiscrete metrizable topological group, for all $n \geq 2$.

Cuchillo-Ibanez et al. [10] introduced a topology on the set of shape morphisms between arbitrary topological spaces X, Y, $Sh(X,Y)$. Moszyńska [21] showed that for a compact Hausdorff space (X,x), the kth shape group $\check{\pi}_k(X,x)$, $k \in \mathbb{N}$, is isomorphic to the set $Sh((S^k,*),(X,x))$ and Bilan [2] mentioned that the result can be extended for all topological spaces. The authors [22], considering the latter topology on the set of shape morphisms between pointed spaces, obtained a topology on the shape homotopy groups of arbitrary spaces, denoted by $\check{\pi}_k^{top}(X,x)$ and showed that with this topology, the kth shape group $\check{\pi}_k^{top}(X,x)$ is a Hausdorff topological group, for all $k \in \mathbb{N}$. Moreover, they obtained some topological properties of these groups under some conditions such as movability, $\mathbb{N}$-compactness and compactness. In particular, they proved that $\check{\pi}_k^{top}$ commutes with finite product of compact Hausdorff spaces. Also, they presented two spaces X and Y with the same shape homotopy groups such that their topological shape homotopy groups are not isomorphic.

The aim of this paper is to introduce a topology on the coarse shape homotopy groups $\check{\pi}_k^*(X,x)$ and to provide some topological properties of these groups. First, similarly to the techniques in [10], we introduce a topology on the set of coarse shape morphisms $Sh^*(X,Y)$, for arbitrary topological spaces X and Y. Several properties of this topology such as continuity of the map $\Omega : Sh^*(X,Y) \times Sh^*(Y,Z) \longrightarrow Sh^*(X,Z)$ given by the composition $\Omega(F^*,G^*) = G^* \circ F^*$ and the equality $Sh^*(X,Y) = \varprojlim Sh^*(X,Y_\mu)$, for an HPol-expansion $\mathbf{q} : Y \to (Y_\mu, q_{\mu\mu'}, M)$ of Y, are proved which are useful to hereinafter results. Moreover, we show that this topology can also be induced from an ultrametric similarly to the process in [9].

By the above topology, we can consider a topology on the coarse shape homotopy group $\check{\pi}_k^{*^{top}}(X,x) = Sh^*((S^k,*),(X,x))$ which makes it a Hausdorff topological group, for all $k \in \mathbb{N}$ and any pointed topological space (X,x). It is known that if X and Y are compact Hausdorff spaces, then $X \times Y$ is a product in the coarse shape category [23, Theorem 2.2]. In this case, we show that the kth topological coarse shape group commutes with finite product, for all $k \in \mathbb{N}$. Also, we prove that movability of $\check{\pi}_k^{*^{top}}(X,x)$ can be concluded from the movability of (X,x), for topological space (X,x) with some conditions. As previously mentioned, $\check{\pi}_k(X,x)$ with the topology defined by Cuchillo-Ibanez et al. [10] on the set of shape morphisms, is a topological group. We show that this topology also coincides with the topology induced by $\check{\pi}_k^{*^{top}}(X,x)$ on the subspace $\check{\pi}_k(X,x)$.

2. Preliminaries

Recall from [1] some of the main notions concerning the coarse shape category and pro*-category. Let $\mathcal{T}$ be a category and let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be two inverse systems in the category $\mathcal{T}$. An *S^*-morphism* of inverse systems, $(f, f_\mu^n) : \mathbf{X} \to \mathbf{Y}$, consists of an index function $f : M \to \Lambda$ and of a set of $\mathcal{T}$-morphisms $f_\mu^n : X_{f(\mu)} \to Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$, such that for every related pair $\mu \leq \mu'$ in M, there exist a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f(\mu')$, and an $n \in \mathbb{N}$ so that for every $n' \geq n$,

$$q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda} = f_\mu^{n'} p_{f(\mu)\lambda}.$$

If $M = \Lambda$ and $f = 1_\Lambda$, then $(1_\lambda, f_\lambda^n)$ is said to be a *level S^*-morphism*. The *composition* of S*-morphisms $(f, f_\mu^n) : \mathbf{X} \to \mathbf{Y}$ and $(g, g_\nu^n) : \mathbf{Y} \to \mathbf{Z} = (Z_\nu, r_{\nu\nu'}, N)$ is an S*-morphism $(h, h_\nu^n) = (g, g_\nu^n)(f, f_\mu^n) : \mathbf{X} \to \mathbf{Z}$, where $h = fg$ and $h_\nu^n = g_\nu^n f_{g(\nu)}^n$, for all $n \in \mathbb{N}$. The *identity S^*-morphism* on $\mathbf{X}$ is an S*-morphism $(1_\Lambda, 1_{X_\lambda}^n) : \mathbf{X} \to \mathbf{X}$, where 1_Λ is the identity function and $1_{X_\lambda}^n = 1_{X_\lambda}$ in $\mathcal{T}$, for all $n \in \mathbb{N}$ and every $\lambda \in \Lambda$.