Topology and its Applications 219 (2017) 29–42

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Transitive and series transitive maps on $\mathbb{R}^d$

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A R T I C L E I N F O

Article history:
Received 18 November 2016
Received in revised form 10 January 2017
Accepted 12 January 2017
Available online 16 January 2017

MSC:
54H20

Keywords:
Transitivity
Series transitivity
Mixing
Euclidean maps
Polish group
Hypercyclic operator

A B S T R A C T

Motivated by the behavior of topologically transitive homomorphisms of Polish abelian groups, we say a continuous map $f : \mathbb{R}^d \to \mathbb{R}^d$ is 'series transitive' if for any two nonempty open sets $U, V \subset \mathbb{R}^d$, there exist $x \in U$ and $n \in \mathbb{N}$ such that $\sum_{j=0}^{n-1} f^j(x) \in V$. We show that any map on a discrete and closed subset of $\mathbb{R}^d$ can be extended to a mixing map of $\mathbb{R}^d$, and use this result to produce a mixing map $f : \mathbb{R}^d \to \mathbb{R}^d$ (for each $d \in \mathbb{N}$) which is also series transitive. We have examples to say that transitivity and series transitivity are independent properties for continuous self-maps of $\mathbb{R}^d$. We also construct a chaotic map (i.e., a transitive map with a dense set of periodic points) $f : \mathbb{R}^d \to \mathbb{R}^d$ such that f is arbitrarily close to and asymptotic to the identity map. Finally, we make a few observations about topological transitivity of continuous homomorphisms of Polish abelian groups.

1. Introduction

Our work is motivated by the following observation.

Proposition 1. *Let X be a Polish abelian group without isolated points, and $f : X \to X$ be a continuous group homomorphism. If f is topologically transitive, then for any two nonempty open sets $U, V \subset X$, there exist $x \in U$ and $n \in \mathbb{N}$ such that $\sum_{j=0}^{n-1} f^j(x) \in V$.*

The proof of Proposition 1 will be supplied in the final section.

A continuous map (not necessarily a group homomorphism) on a Polish abelian group satisfying the conclusion of Proposition 1 will be called a *series transitive* map. In this article, we investigate the relation between transitivity and series transitivity for continuous self-maps of $\mathbb{R}^d$ for $d \in \mathbb{N}$.

It is rather easy to construct (topologically) transitive maps on $[0, 1]$, a simple example being the tent map $x \mapsto 1 - |2x - 1|$. The theory of transitive maps on $[0, 1]$ is reasonably well-understood and is covered

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http://dx.doi.org/10.1016/j.topol.2017.01.008

in the books [5,7,15,14,16,24]. However, producing transitive maps on $\mathbb{R}^d$ for $d \geq 1$ is not quite trivial. In fact, this topic has a long history. Possibly the first example of a transitive map of the plane $\mathbb{R}^2$ was given by Schnirelmann [26]. Following that, Besicovitch [9,10] constructed such maps on the plane in a simpler way. Oxtoby [19] used category techniques to show that measure preserving transitive homeomorphisms are residual (comeager) in the collection of all measure preserving homeomorphisms of $[0,1]^d$ for $d \geq 2$, and deduced the existence of transitive homeomorphisms on $\mathbb{R}^d$ for $d \geq 2$ (see also Chapter 18 of [20]). By modifying Schnirelman–Besicovitch maps, Sidorov [25] produced special types of transitive homeomorphisms on $\mathbb{R}^d$ for every $d \geq 2$. A corollary of the work of Cairns and Kolganova [11] is that every compact triangulable manifold of dimension ≥ 2 admits a *chaotic* (i.e., transitive with a dense set of periodic points) homeomorphism, from which it can also be deduced that $\mathbb{R}^d$ admits chaotic homeomorphisms for $d \geq 2$. Alpern and Prasad [2] lifted toral homeomorphisms to produce chaotic homeomorphisms that are also volume preserving on $\mathbb{R}^d$ for $d \geq 2$. Shortly afterwards, they also showed [3] that the collection of chaotic homeomorphisms is dense in the collection of all measure preserving homeomorphisms of $\mathbb{R}^d$ for $d \geq 2$. Nagar and Sesha Sai [18] constructed transitive maps on $\mathbb{R}$. Cairns, Jessup and Nicolau [12] studied transitive homeomorphisms on quotients of tori, and in particular they pointed out a somewhat simple construction of chaotic homeomorphisms on $\mathbb{R}^2$.

For analogous results in Ergodic Theory, we refer the reader to the survey [4]. We do not consider transitive maps on compact connected manifolds in this article, but the interested reader may refer, for instance, to [21,6,11,1,13].

In contrast to the approaches of some of the references mentioned above, the constructions in this article are relatively elementary: we do not use any Measure Theory, and we do not need the theory of maps on the torus. We also remark that Proposition 1 is not useful on $\mathbb{R}^d$ because any continuous group homomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ must be linear and it is known (see Proposition 1.1 of [8]) that a linear map of $\mathbb{R}^d$ cannot be topologically transitive.

In section 3, we will show that any map defined on a discrete and closed subset of $\mathbb{R}^d$ can be extended to a (topologically) mixing map of $\mathbb{R}^d$ in such a way that every point has a dense backward orbit, and then will use this fact to construct an example of a mixing map $f : \mathbb{R}^d \to \mathbb{R}^d$ which is also series transitive. The technique behind the proof of the extension result may be roughly summarized as follows: (i) write $\mathbb{R}^d$ as a union $\mathbb{R}^d = \bigcup_{j=1}^{\infty} K_j$ of suitable compact sets K_j with pairwise disjoint interiors, (ii) for each $j \in \mathbb{N}$, define $f_j : K_j \to \mathbb{R}^d$ having a certain expansion property such that f_j is identity on the boundary ∂K_j of K_j, (iii) paste together f_j's to obtain a global map of the required type. We remark that the idea that a 'piecewise expanding map' is a good candidate for a topologically transitive map, is not new; see for instance [22,18].

In section 4, we will produce examples to say that topological transitivity and series transitivity are independent properties for continuous maps $f : \mathbb{R}^d \to \mathbb{R}^d$. In section 5, we will construct a chaotic map $f : \mathbb{R}^d \to \mathbb{R}^d$ such that f is arbitrarily close to and asymptotic to the identity map. In the final section, we will make a few more observations regarding continuous homomorphisms of Polish abelian groups, just to provide a contrast. In particular, we will establish that transitivity and series transitivity are equivalent for continuous linear operators of separable Fréchet spaces.

2. Preliminaries

For the purpose of this article, by a *dynamical system* we mean a pair (X, f), where X is a Polish abelian group (i.e., a complete separable metric space having the structure of an abelian topological group) without isolated points, and $f : X \to X$ is a continuous map (not necessarily a group homomorphism). If (X, f) is a dynamical system, the f-orbit $O(f, x)$ of an element $x \in X$ is defined as $O(f, x) = \{x, f(x), f^2(x), \ldots\}$, where f^n stands for the n-fold self-composition of f. The backward orbit $BO(f, x)$ of an element $x \in X$ is defined as $BO(f, x) = \{z \in X : f^n(z) = x \text{ for some integer } n \geq 0\}$. We say $x \in X$ is a *periodic point*