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A natural generalization of regular convex polyhedra ☆

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ABSTRACT

As a natural generalization of surfaces of Platonic solids, we define a class of polyhedra, called simple regular polyhedral BP-complexes, as a class of 2-dimensional polyhedral metric complexes satisfying certain conditions on their vertex sets, and we give a complete classification of such polyhedra. They are either the surface of a Platonic solid, a p-dodecahedron, a p-icosahedron, an m-covered regular n-gon for some $m \geqq 2$ or a complete tripartite polygon.

1. Introduction

The objects studied in this paper are 2-dimensional polyhedral metric complexes obtained by identifying isometric edges of convex polygons. We focus on their geometric aspects as geodesic metric spaces, in which their cell complex structures play auxiliary roles.

Let X be a connected 2-dimensional locally finite cell complex. By a cell of X we always mean an open cell. We call X *homogeneous* if each 1-cell of X is a proper face of some 2-cell of X. By taking an appropriate subdivision if necessary, we may assume that every homogeneous 2-dimensional cell complex X in question satisfies the following condition:

($*$) The number of vertices on the boundary of each 2-cell is at least 3.

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A 2-dimensional homogeneous locally finite cell complex X satisfying $(*)$ is called a *polyhedral complex* if it is equipped with a piecewise linear metric d satisfying the following property: if e is a 2-cell of X whose boundary has n vertices ($n \geqq 3$), then the closure $\bar{e}$ of e is isometric to some n-gon in the Euclidean plane $\mathbf{R}^2$ with corresponding vertices, with respect to the intrinsic metric induced by d.

For a polyhedral complex X, we call the closure of a 2-cell a face, the closure of a 1-cell an edge, and a 0-cell a vertex, respectively. The metric space X is an M_0-polyhedral complex in the sense of [2], but is not necessarily a CAT(0) space.

A polyhedral complex X is said to be *non-degenerate* if, for any two distinct faces E_1 and E_2 of X, the intersection $E_1 \cap E_2$ is either an edge, a vertex or an empty set. Otherwise, we call X *degenerate*. For instance, let S be an n-gon for $n \geqq 3$, and let S_1 and S_2 be its isometric copies. Identifying the corresponding edges of S_1 and S_2, we obtain a degenerate polyhedral complex X, which is called the *double* n-gon. Clearly, no degenerate polyhedral complex can be isometrically embedded into Euclidean spaces of any dimension in such a way that each face is embedded in some plane.

Recall that every convex polyhedron P is defined as the surface of a finite intersection of half-spaces in the Euclidean 3-space $\mathbf{R}^3$. Note that an intrinsic metric on P is induced from $\mathbf{R}^3$. Disregarding the extrinsic geometry, we consider a convex polyhedron P as a non-degenerate polyhedral complex satisfying the following topological condition (T) and the convexity condition (C):

(T) P is homeomorphic to a 2-sphere S^2.
(C) The sum of interior angles at any vertex v is less than 2π.

Also, by a regular convex polyhedron we mean a non-degenerate convex polyhedron P satisfying the following regularity conditions (R1) and (R2):

(R1) There is a regular polygon to which every face is congruent.
(R2) The space of directions at every vertex is isometric to each other.

The definition of the space of directions is given in Section 2. As is well known, every regular non-degenerate convex polyhedron is the surface of a Platonic solid (namely, a tetrahedron, a cube, an octahedron, a dodecahedron or an icosahedron). For any of such polyhedron, three or more faces meet at each vertex. On the other hand, if only two faces meet at each vertex, then it is degenerate and hence is a double regular polygon. It is convenient to regard a double regular polygon as a regular convex polyhedron.

The purpose of this paper is to classify all polyhedra in a class of polyhedral complexes satisfying some conditions as a generalization of regular polyhedra. Our primary concern is geometry of piecewise Riemannian spaces which are not manifolds. We are also concerned with their model spaces.

To this end, we examine the conditions (T) and (C). First, we assume that a polyhedral complex P satisfies

(I) The intersection of any two faces is connected.

Such a polyhedral complex is said to be *simple*. Obviously, regular polyhedra and non-degenerate polyhedral complexes satisfy this condition.

Instead of the condition (T), we consider a weaker condition

(T′) P is a 2-manifold.