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$\frac{1}{3}$-homogeneous dendrites

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A *continuum* is a nondegenerate compact connected metric space. A *dendrite* is a locally connected continuum containing no simple closed curves. A continuum X is said to be $\frac{1}{3}$*-homogeneous* if there exist three nonempty and mutually disjoint subsets O_1, O_2 and O_3 of X such that $X = O_1 \cup O_2 \cup O_3$ and for each $x, y \in X$ there exists a homeomorphism $h\colon X \to X$ such that $h(x) = y$ if and only if $x, y \in O_i$ for some $i \in \{1, 2, 3\}$. In 2006 V. Neumann-Lara, P. Pellicer-Covarrubias, and I. Puga showed that a dendrite X is $\frac{1}{2}$-homogeneous if and only if X is an arc. The purpose of this paper is to extend this result and classify all $\frac{1}{3}$-homogeneous dendrites.

1. Introduction

For a topological space X, we denote by $\mathcal{H}(X)$ the group of homeomorphisms of X onto itself. For $n \in \mathbb{N}$, a space X is said to be $\frac{1}{n}$*-homogeneous* provided that there are exactly n orbits for the action of $\mathcal{H}(X)$ on X. Given $x \in X$, the set $\mathrm{Orb}_X(x) = \{h(x)\colon h \in \mathcal{H}(X)\}$ is called the *orbit of* x *in* X. More general, a nonempty subset O of X is said to be an *orbit of* X if there is $x \in X$ such that $O = \mathrm{Orb}_X(x)$. If O is an orbit of X, then $y, z \in O$ if and only if there exists a homeomorphism $h\colon X \to X$ such that $h(y) = z$. It is not difficult to see that the collection $\mathfrak{R}(X) = \{\mathrm{Orb}_X(x)\colon x \in X\}$ is a partition of X. Hence, for $n \in \mathbb{N}$, X is $\frac{1}{n}$-homogeneous if and only if the cardinality of the collection $\mathfrak{R}(X)$ is exactly n. Such natural number n is also called the *degree of homogeneity* of X.

A *continuum* is a nondegenerate compact connected metric space. An *arc* is a space homeomorphic to the interval $[0, 1]$. A *simple closed curve* is a space homeomorphic to the unit circle S^1 in $\mathbb{R}^2$. A *Cantor set* is a

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space homeomorphic to the standard middle-third Cantor set. A *dendrite* is a locally connected continuum containing no simple closed curves.

For $n \in \mathbb{N} - \{1\}$, the study of $\frac{1}{n}$-homogeneous continua formally started in 1989 with the work done by H. Patkowska in [21], in which the term $\frac{1}{n}$-homogeneous is coined and $\frac{1}{2}$-homogeneous polyhedra are classified. Without an explicit use of such term, in 1969 J. Krasinkiewicz proved in [11] that the universal Sierpiński curve is $\frac{1}{2}$-homogeneous. From 2006 to recent dates new results concerning $\frac{1}{n}$-homogeneous continua have appeared in the literature, most of them involving the research of P. Pellicer-Covarrubias either alone or in collaboration with other researchers (see [10,14,15,17–20,22,23] and [24]). Some other papers dealing with $\frac{1}{n}$-homogeneous continua are [3–5] and [6].

In [20, Lemma 3.5] it is shown that a dendrite X is $\frac{1}{2}$-homogeneous if and only if X is an arc. The purpose of this paper is to extend this result and classify $\frac{1}{3}$-homogeneous dendrites (see Theorem 7.16).

The paper is divided into seven sections. After this Introduction, in Section 2 we present some notions, notation and general results that we will use in the rest of the paper. In Section 3 we present some properties of dendrites that we require for the classification of $\frac{1}{3}$-homogeneous dendrites. In Section 4 we present some results that involve $\frac{1}{3}$-homogeneous dendrites. In this section we show that a $\frac{1}{3}$-homogeneous dendrite has either one ramification point or infinitely many ramification points (see Theorem 4.2). We also show, among other results, that all dendrites with only one ramification point are $\frac{1}{3}$-homogeneous. In Section 5 we classify all $\frac{1}{3}$-homogeneous dendrites without free arcs (see Theorem 5.3). In Section 6 we classify all $\frac{1}{3}$-homogeneous dendrites with free arcs, infinitely many ramification points and whose set of end points is closed (see Theorem 6.6). Finally, in Section 7, we classify all $\frac{1}{3}$-homogeneous dendrites with free arcs, infinitely many ramification points and whose set of end points is not closed (see Theorem 7.15). In this way we obtain the classification of all $\frac{1}{3}$-homogeneous dendrites (see Theorem 7.16). For notions not defined here we refer the reader to [9].

2. General notions

For a topological space X and $A \subset X$, the symbols $\mathrm{Cl}_X(A)$, $\mathrm{Int}_X(A)$ and $\mathrm{Bd}_X(A)$ denote the closure, the interior and the boundary of A in X, respectively. If a sequence $\{x_n\}_n$ of points of X converges to an element $x \in X$, we write either $x_n \to x$ or $\lim_{n\to\infty} x_n = x$. The cardinality of A is denoted by $|A|$ and its diameter by $\mathrm{diam}(A)$. For a set Y, the identity function on Y is denoted by 1_Y.

A topological space X is said to be *σ-connected* if X cannot be written as the union of more than one and at most countably infinitely many nonempty, mutually disjoint, closed subsets.

A *finite graph* is a continuum which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect in one or both of their end points. A *tree* is a finite graph that contains no simple closed curves.

Let X be a continuum and $p \in X$. We say that p is a *cut point* of X if the set $X - \{p\}$ is not connected. If β is a cardinal number, then we say that *p is of order less than or equal to β* or that *p has order less than or equal to β*, written $\mathrm{ord}_X(p) \leq \beta$, if for each open neighborhood U of p there exists an open neighborhood V of p such that $V \subset U$ and $|\mathrm{Bd}_X(V)| \leq \beta$. We say that *$p$ is of order β* or that *p has order β*, written $\mathrm{ord}_X(p) = \beta$, if $\mathrm{ord}_X(p) \leq \beta$ and for each cardinal number α so that $\alpha < \beta$, it follows that $\mathrm{ord}_X(p) \nleq \alpha$. We say that p is an *end point* of X if $\mathrm{ord}_X(p) = 1$, a *ramification point* of X if $\mathrm{ord}_X(p) > 2$ and an *ordinary point* if $\mathrm{ord}_X(p) = 2$. We denote by $E(X)$, $R(X)$, $O(X)$ and $\mathrm{Cut}(X)$ the set of end points of X, the set of ramification points of X, the set of ordinary points of X and the set of cut points of X, respectively. If X is an arc and $f\colon [0,1] \to X$ is a homeomorphism, then $E(X) = \{f(0), f(1)\}$.

If X and Y are continua and $h\colon X \to Y$ is a homeomorphism then, for each $p \in X$ we have $\mathrm{ord}_X(p) = \mathrm{ord}_Y(h(p))$. Hence $h(E(X)) = E(Y)$, $h(O(X)) = O(Y)$ and $h(R(X)) = R(Y)$. This implies that if the sets $E(X), O(X)$ and $R(X)$ are nonempty, then the degree of homogeneity of X is greater than or equal to three.