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Generalized recurrence and the nonwandering set for products ☆

Jim Wiseman

Agnes Scott College, Decatur, GA 30030, United States

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ABSTRACT

For continuous maps of compact metric spaces $f : X \to X$ and $g : Y \to Y$ and for various notions of topological recurrence, we study the relationship between recurrence for f and g and recurrence for the product map $f \times g : X \times Y \to X \times Y$. For the generalized recurrent set GR, we see that $\mathrm{GR}(f \times g) = \mathrm{GR}(f) \times \mathrm{GR}(g)$. For the nonwandering set NW, we see that $\mathrm{NW}(f \times g) \subset \mathrm{NW}(f) \times \mathrm{NW}(g)$ and give necessary and sufficient conditions on f for equality for every g. We also consider product recurrence for the chain recurrent set, the strong chain recurrent set, and the Mañé set.

1. Introduction

Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact metric spaces. We are interested in the relationship between recurrence for f and g and recurrence for the product map $f \times g : X \times Y \to X \times Y$, and in how that relationship varies depending on which notion of recurrence we consider.

The strongest notion of recurrence is periodicity. It is clear that $\mathrm{Per}(f \times g)$, the set of periodic points for $f \times g$, is equal to $\mathrm{Per}(f) \times \mathrm{Per}(g)$. A slightly weaker condition is that a point is *(positively) recurrent* if it is in its own ω-limit set. The question of whether the positive recurrent set of a given product is equal to the product of the positive recurrent sets has been well studied and has led to some very deep and interesting mathematics; see [2] and [13] and the references therein. In this paper, we consider the corresponding question for several less restrictive notions of recurrent set, most importantly the generalized recurrent set and the nonwandering set.

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E-mail address: jwiseman@agnesscott.edu.

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The interesting dynamics occurs on the nonwandering set, so in order to understand the relationship between the dynamics of a product map and the dynamics of the original maps, we need to understand the nonwandering set; we give necessary and sufficient conditions (Theorem 3.12) for a point $x \in X$ to be product nonwandering, that is, for (x, y) to be nonwandering for $f \times g$ for any g and any nonwandering point $y \in Y$. Auslander's generalized recurrent set $\mathrm{GR}(f)$ (defined originally for flows (see [4]), and extended to maps (see [1,3])) is a larger and in many ways more dynamically natural set, particularly for understanding Lyapunov functions; see [8] and the references in [15]. We show that $\mathrm{GR}(f \times g) = \mathrm{GR}(f) \times \mathrm{GR}(g)$ (Theorem 3.1). The same is clearly true for the chain recurrent set, which reflects a still broader notion of recurrence.

We also consider product recurrence for Easton's strong chain recurrent set and Fathi and Pageault's Mañé set. These results come up for the most part in the study of the generalized recurrent set, but are also of independent interest.

In section 2, we give definitions and background information for the various notions of recurrence and for metrics on the product space. In section 3, we state and prove our results.

I am grateful to the referee for useful suggestions to improve and clarify this work.

2. Definitions and background

Throughout the paper, let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact metrizable spaces; unless stated otherwise, we will use the metrics d_X and d_Y respectively. Let $B_d(x; \varepsilon)$ be the closed ε-ball around x, $B_d(x; \varepsilon) = \{x' \in X : d(x, x') \leq \varepsilon\}$.

2.1. Recurrent sets

Definition 2.1. A point $x \in X$ is *nonwandering* for f if for any neighborhood U of X, there exists an $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. We denote by $\mathrm{NW}(f)$ the set of nonwandering points.

Definition 2.2. An (ε, f, d_X)-*chain* (or (ε, d_X)-*chain*, if it is clear what the map is, or ε-*chain*, if the metric is also clear) of length n from x to x' is a sequence $(x = x_0, x_1, \dots, x_n = x')$ such that $d_X(f(x_{i-1}), x_i) \leq \varepsilon$ for $i = 1, \dots, n$. A point x is *chain recurrent* if for every $\varepsilon > 0$, there is an ε-chain from x to itself. We denote by $\mathrm{CR}(f)$ the set of chain recurrent points. (Chain recurrence is independent of the choice of metric; see, for example, [9].)

The following definition is due to Easton [7].

Definition 2.3. A *strong* (ε, f, d_X)-*chain* (or *strong* (ε, d_X)-*chain* or *strong* ε-*chain*) from x to x' is a sequence $(x = x_0, x_1, \dots, x_n = x')$ such that the sum of the errors is bounded by ε, that is, $\sum_{i=1}^{n} d_X(f(x_{i-1}), x_i) \leq \varepsilon$. A point x is d_X-*strong chain recurrent* (or *strong chain recurrent*) if for every $\varepsilon > 0$, there is a strong (ε, d_X)-chain from x to itself. We denote the set of strong chain recurrent points by $\mathrm{SCR}_{d_X}(f)$.

We write $x_1 \sim_{d_X} x_2$ if for any $\varepsilon > 0$ there is a strong (ε, f, d_X)-chain from x_1 to x_2 and one from x_2 to x_1; then $\sim_{d_X}$ is a closed equivalence relation on $\mathrm{SCR}_{d_X}(f)$, and each equivalence class is a closed invariant set.

The strong chain recurrent set does depend on the choice of metric; see, for example, [16]. (Note, however, that it does not change if the new metric is bi-Lipschitz equivalent to the original [16, Prop. 3.2].) One way to eliminate this dependence is to take the intersection over all possible choices. This leads to the following definition.

Definition 2.4 *([8])*. The *generalized recurrent set* $\mathrm{GR}(f)$ is $\bigcap_{d'_X} \mathrm{SCR}_{d'_X}(f)$, where the intersection is over all metrics d'_X compatible with the topology of X.