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On variants of the principle of consistent choices, the minimal cover property and the 2-compactness of generalized Cantor cubes

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A B S T R A C T

In set theory without the Axiom of Choice (AC), we study the deductive strength of variants of the Principle of Consistent Choices (PCC) and their relationship with the minimal cover property, the 2-compactness of generalized Cantor cubes, and with certain weak choice principles. (Complete definitions are given in Section "Notation and terminology".) Among other results, we establish the following:

1. "For every infinite set X, the generalized Cantor cube 2^X has the minimal cover property" (MCP) implies "PCC restricted to families of 2-element sets" (F_2), which in turn implies "for every infinite set X, 2^X is 2-compact" ($Q(2)$). Moreover, 'MCP implies F_2' is not reversible in ZFA (i.e., ZF, the Zermelo–Fraenkel set theory minus AC, with the Axiom of Extensionality weakened in order to permit the existence of atoms).
   The above results *strengthen* related results in Howard–Tachtsis "On the minimal cover property in **ZF**" and "On the set-theoretic strength of the n-compactness of generalized Cantor cubes".
2. "Every Dedekind-finite set is finite" (DF = F) implies the Principle of Partial Consistent Choices (PPCC) – the latter principle being *introduced here* – which in turn implies $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$ (i.e., the axiom of choice for denumerable families of non-empty finite sets). None of the previous implications is reversible in ZF.
3. The Principle of Countable Consistent Choices ($\mathsf{PCC}^{\aleph_0}$), which is *introduced here*, is equivalent to $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$.
4. Rado's Lemma (RL) + "every infinite set has an infinite linearly orderable subset" implies PPCC. In addition, RL does not imply PPCC in ZFA, PPCC does not imply RL in ZF, and PPCC does not imply "every infinite set has an infinite linearly orderable subset" in ZFA.
5. PPCC does not imply "there are no amorphous sets" in ZFA.
6. F_2 implies "there are no amorphous sets" and the implication is not reversible in ZF. This *clarifies* the relationship between the latter two statements, whose status is mentioned as *unknown* in Howard–Rubin "Consequences of the Axiom of Choice".
7. "Every infinite partially ordered set has either an infinite chain or an infinite antichain" (CAC) does not imply X, where $\mathsf{X} \in \{\mathsf{PPCC}, \mathsf{F}_2\}$, in ZF. In addition, F_2 does not imply CAC in ZFA.

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1. Notation and terminology

Below, we list the definitions of notions and principles, as well as their notations, which shall be used in the sequel.

Definition 1. A binary relation $\leq$ on a set P is called a *partial order* on P if $\leq$ is reflexive, anti-symmetric, and transitive. The ordered pair $(P, \leq)$ is called a *partially ordered set*, or a partial order, or simply a *poset*.

A *linearly ordered set* is a poset $(P, \leq)$ such that $\forall p \in P$, $\forall q \in P$, $p \leq q$ or $q \leq p$.

Let $(P, \leq)$ be a poset.

(a) A subset $C \subseteq P$ is called a *chain* in P if $(C, \leq\restriction C)$ is linearly ordered.

(b) A subset $A \subseteq P$ is called an *antichain* in P if any two distinct elements $a, b \in A$ are incomparable, i.e., $a \not\leq b$ and $b \not\leq a$.

Definition 2.

1. A set X is called *Dedekind-finite* if there is no one-to-one mapping $f : \omega \to X$. Equivalently, X is Dedekind-finite if there is no one-to-one mapping from X into a proper subset of X. If X is not Dedekind-finite, then X is called *Dedekind-infinite*.
2. An infinite set X (i.e., there is no bijection $f : X \to n$ for any natural number n) is called *amorphous* if X cannot be expressed as a disjoint union of two infinite subsets.
3. If X is a set and $n \in \omega$, then $[X]^n$ is the set of all n-element subsets of X and $[X]^{<\omega} = \bigcup\{[X]^k : k \in \omega\}$ is the set of all finite subsets of X.

Definition 3. Let $(X_i, T_i)_{i\in I}$ be an infinite family of topological spaces and let $X = \prod_{i\in I} X_i$ be the Tychonoff product of the X_i's. Let $n \in \omega \setminus \{0\}$.

A basic open set U of X is called *n-basic open set* if there exists a set $J \in [I]^n$ and open sets U_j in X_j, $j \in J$, such that $U = \prod_{j\in J} U_j \times \prod_{i\in I\setminus J} X_i$. The complement of an n-basic open set is called an *n-basic closed set*.

X is called *n-compact* if every cover $\mathcal{U}$ consisting of n-basic open sets has a finite subcover. Equivalently, X is n-compact if every family of n-basic closed sets with the finite intersection property (fip) has a non-empty intersection.

For all $i \in I$, let $X_i = 2$ $(= \{0, 1\})$ and let T_i be the discrete topology on X_i. Adopting the terminology of [11], the Tychonoff product 2^I is called a *generalized Cantor cube*. The collection $\mathcal{B}_I = \{[p] : p \in \mathrm{Fn}(I, 2)\}$, where $\mathrm{Fn}(I, 2)$ is the set of all finite partial functions from I into 2, i.e., $\mathrm{Fn}(I, 2) = \{p$: p is a function, $\mathrm{dom}(p) \in [I]^{<\omega}$, $\mathrm{ran}(p) \subseteq 2\}$, and $[p] = \{f \in 2^I : p \subset f\}$, is the standard clopen (i.e., simultaneously closed and open) base for the product topology on 2^I. For every $k \in \omega \setminus \{0\}$, let $\mathcal{B}_I^k = \{[p] \in \mathcal{B}_I : |p| = k\}$. According to the above terminology, the elements of $\mathcal{B}_I^k$, $k \in \omega \setminus \{0\}$, are k-basic clopen sets of 2^X. For simplicity, we sometimes denote a 1-basic clopen set $[\{(i, \alpha)\}]$, $i \in I$, $\alpha \in 2$, by $\langle i, \alpha\rangle$ (the latter notation for 1-basic clopen sets has also been used in [12]).

It is clear that $\mathcal{B}_I = \bigcup\{\mathcal{B}_I^k : k \in \omega \setminus \{0\}\}$ and that 2^I is k-compact if and only if every cover $\mathcal{U} \subset \mathcal{B}_I^k$ of 2^I has a finite subcover.

Definition 4. For each $n \in \omega \setminus \{0\}$, $Q(n)$ denotes the statement "for every infinite set I, the generalized Cantor cube 2^I is n-compact". The principle $Q(n)$ was introduced in Keremedis–Tachtsis [14].

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