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On the 3-dimensional invariant for cyclic contact branched coverings

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ABSTRACT

We give a formula of the 3-dimensional invariant for a cyclic contact branched covering of the standard contact S^3.

1. Introduction

Let $\widetilde{M} \to M$ be a branched covering of a 3-manifold M, branched along a link $K \subset M$. When M has a contact structure ξ and K is a transverse link in the contact 3-manifold (M, ξ), $\widetilde{M}$ has a contact structure $\widetilde{\xi}$ which is a perturbation of the pull-back $\pi^*\xi$. Such a contact structure is unique up to isotopy, and we call the contact 3-manifold $(\widetilde{M}, \widetilde{\xi})$ the *contact branched covering* of (M, ξ), branched along the transverse link K.

Let (M, ξ) be a p-fold cyclic contact branched covering of (S^3, ξ_{std}) (the standard contact S^3), branched along a transverse link K. In [5, Theorem 1.4], it is shown that the Euler class $e(\xi)$ is zero, and the 3-dimensional invariant $d_3(\xi) \in \mathbb{Q}$ (see [3] for definition) only depends on a topological link type of K and its self-linking number. However, no explicit formula of $d_3(\xi)$ had been given and it is not an easy task to compute $d_3(\xi)$ when p is large or K is complicated.

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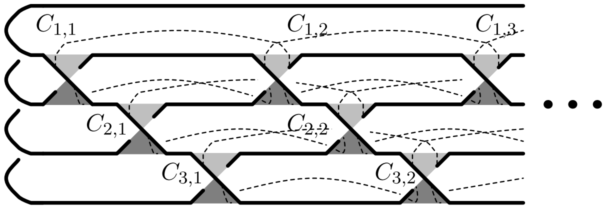

Fig. 1. Page S of the open book (S, ψ) inside S^3.

In this note, we show a direct formula of $d_3(\xi)$ in terms of its branch locus K.

Theorem 1.1. *If a contact 3-manifold (M, ξ) is a p-fold cyclic contact branched covering of (S^3, ξ_{std}), branched along a transverse link K, then*

$$d_3(\xi) = -\frac{3}{4} \sum_{\omega:\omega^p=1} \sigma_\omega(K) - \frac{p-1}{2} sl(K) - \frac{1}{2}p.$$

Here $\sigma_\omega(K)$ denotes the Tristram–Levine signature, the signature of $(1-\omega)A + (1-\overline{\omega})A^T$, where A denotes the Seifert matrix for K, and $sl(K)$ denotes the self-linking number.

Thus, our formula tells us that $d_3(\xi)$ actually only depends on the concordance class of K and the self-linking number. By the slice Bennequin inequality [7], it also shows that the smooth 4-genus $g_4(K)$ of K gives a lower bound of $d_3(\xi)$.

Corollary 1.2. *If a contact 3-manifold (M, ξ) is a p-fold cyclic contact branched covering of (S^3, ξ_{std}) branched along K, then $d_3(\xi) \geq -\frac{5}{2}(p-1)g_4(K) - \frac{1}{2}$.*

2. Proof

Proof of Theorem 1.1. Let (M, ξ) be a p-fold cyclic contact branched covering, branched along a transverse link K in (S, ξ_{std}). We put the transverse link K as a closed braid, the closure of an m-braid α (with respect to the disk open book decomposition for (S^3, ξ_{std})).

Let (S, ψ) be the open book decomposition of (S^3, ξ_{std}), whose binding is the (p, m)-torus link. Inside S^3, the page S is an obvious Seifert surface of the (p, m)-torus link which we view as the closure of the m-braid $(\sigma_1 \cdots \sigma_{m-1})^p$ as we illustrate in Fig. 1.

Topologically, the page S is the p-fold cyclic branched covering of the disk D^2, branched along m-points. Let $\pi : B_m = MCG(D^2 \setminus \{m \text{ points}\}) \to MCG(S)$ be the map induced by the branched covering map, which is written by $\pi(\sigma_i) = D_{i,1} \cdots D_{i,p-1}$ [5, Lemma 3.1]. Here $D_{i,j}$ denotes the right-handed Dehn twist along the curve $C_{i,j}$ on S, given in Fig. 1. (Here we are assuming that $MCG(S)$ acts on S from left, so $D_{i,1} \cdots D_{i,p-1}$ means $D_{i,p-1}$ comes first and $D_{i,1}$ last.)

An important observation is that in (S^3, ξ_{std}), the curves $C_{i,j}$ are realized as the Lergendrian unknot with $tb = -1$, $rot = 0$.

By using $D_{i,j}$, the monodromy ψ is written by

$$\psi = \pi(\sigma_{m-1} \cdots \sigma_2\sigma_1) = (D_{m-1,1} \cdots D_{m-1,p-1}) \cdots (D_{2,1} \cdots D_{2,p-1})(D_{1,1} \cdots D_{1,p-1}).$$

Also, $(S, \phi = \pi(\alpha))$ gives an open book decomposition of (M, ξ).

First we draw the surgery diagram of (M, ξ) from its open book decomposition (S, ϕ), following the discussion in [5, Section 3]. We take a factorization of the braid $(\sigma_1^{-1} \cdots \sigma_{m-1}^{-1})\alpha$

$$(\sigma_1^{-1} \cdots \sigma_{m-1}^{-1})\alpha = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_n}^{\varepsilon_n} \qquad (\varepsilon_j \in \{\pm 1\},\ i_j \in \{1, \ldots, m-1\}) \tag{2.1}$$