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Limits of sequences of continuous functions depending on finitely many coordinates

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ABSTRACT

We answer two questions from Bykov (2016) [2] and prove that every Baire one function on a subspace of a countable perfectly normal product is the pointwise limit of a sequence of continuous functions, each depending on finitely many coordinates. It is proved also that a lower semicontinuous function on a subspace of a countable perfectly normal product is the pointwise limit of an increasing sequence of continuous functions, each depending on finitely many coordinates, if and only if the function has a minorant which depends on finitely many coordinates.

1. Introduction

The collection of all continuous maps between topological spaces X and Y is denoted by $\mathrm{C}(X,Y)$. If $A \subseteq Y^X$, then we denote the family of all pointwise limits of sequences of maps from A by $\overline{A}^{\mathrm{p}}$. We say that a subset A of a topological space X is *functionally F_σ (G_δ)*, if it is a union (an intersection) of a sequence of functionally closed (functionally open) sets in X.

Recall that a map $f : X \to Y$ belongs to the *first Borel class*, if the preimage $f^{-1}(V)$ of each open set $V \subseteq Y$ is an F_σ-set in X, and belongs to the *first Baire class*, if $f \in \overline{\mathrm{C}(X,Y)}^{\mathrm{p}}$. We denote the collection of all Borel one maps and Baire one maps between X and Y by $\mathrm{H}_1(X,Y)$ and $\mathrm{B}_1(X,Y)$, respectively. If a Borel (Baire) one map $f : X \to Y$ takes finitely many values, then we say that $f \in \mathrm{H}_1^0(X,Y)$ ($\mathrm{B}_1^0(X,Y)$). Let us observe that $\mathrm{H}_1(X,\mathbb{R}) = \mathrm{B}_1(X,\mathbb{R})$ for a perfectly normal space X (see [8, §31]).

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Properties of functions defined on products of topological spaces are tightly connected with the dependence of such functions on some sets of coordinates. The dependence of continuous mappings on some sets of coordinates was investigated by many mathematicians (see, for instance [9,3,11] and the literature given there). However, the dependence on countably many coordinates serves as a convenient tool for the investigation of Namioka property of separately continuous maps and their analogs [10]. Functions depending locally on finitely many coordinates play an important role in the theory of smoothness and renorming on Banach spaces [4].

Let $(X_n)_{n=1}^{\infty}$ be a sequence of topological spaces, $P = \prod_{n=1}^{\infty} X_n$ and $a = (a_n)_{n\in\mathbb{N}} \in P$ be a point. For every $n \in \mathbb{N}$ and $x \in P$ we put $p_n(x) = (x_1, \dots, x_n, a_{n+1}, a_{n+2}, \dots)$. We say that a set $A \subseteq P$ *depends on finitely many coordinates*, if there exists $n \in \mathbb{N}$ such that for all $x \in A$ and $y \in P$ the equality $p_n(x) = p_n(y)$ implies $y \in A$. A map $f : X \to Y$ defined on a subspace $X \subseteq P$ is *finitely determined* if it *depends on finitely many coordinates*, i.e., there exists $n \in \mathbb{N}$ such that $f(x) = f(y)$ for all $x, y \in X$ with $p_n(x) = p_n(y)$. We denote by $\mathrm{CF}(X, Y)$ the set of all continuous finitely determined maps between X and Y; we write $\mathrm{CF}(X)$ for $\mathrm{CF}(X, \mathbb{R})$.

Vladimir Bykov proved the following results in his recent paper [2].

Theorem A. *Let X be a subspace of a product $P = \prod_{n=1}^{\infty} X_n$ of a sequence of metric spaces X_n. Then*

(1) *every Baire class one function $f : X \to \mathbb{R}$ is the pointwise limit of a sequence of functions in* $\mathrm{CF}(X)$*;*
(2) *a lower semicontinuous function $f : X \to \mathbb{R}$ is the pointwise limit of an increasing sequence of functions from* $\mathrm{CF}(X)$ *if and only if f has a minorant in* $\mathrm{CF}(X)$*.*

The following questions were formulated in [2]:

Question 1. [2, Remark 1] *Does the conclusion (1) of Theorem A holds for an arbitrary completely regular space X?*

Question 2. [2, Remark 2] *Does the conclusion (2) of Theorem A holds for an arbitrary perfectly normal space X?*

We show that the answer on Question 1 is negative (see Theorem 16). We prove that every Borel one function $f : X \to Y$ is a pointwise limit of a sequence of functions from $\mathrm{CF}(X, Y)$ if Y is a path-connected separable metric R-space (in particular, a convex subset of a normed separable space) and one of the following properties holds: X is a Lindelöf subspace of a completely regular product P, or X is a subspace of perfectly normal P (see Theorem 8), or $X = P$ and P is pseudocompact (see Theorem 13). Theorem 9 gives the positive answer on Question 2.

2. Approximation of functions with finitely many values

Let X be a topological space. For a countable family $\mathcal{F} = (f_n : n \in \mathbb{N})$ of continuous functions $f_n : X \to [0, 1]$ we put

$$d_{\mathcal{F}}(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x) - f_n(y)| \tag{2.1}$$

for all $x, y \in X$. Then $d_{\mathcal{F}}$ is a pseudometric on X such that every continuous function $f : (X, d_{\mathcal{F}}) \to Y$ is continuous on X in the initial topology for any topological space Y.