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A note on the characteristic rank of oriented Grassmann manifolds

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We derive new lower bound for the characteristic rank of the canonical 3-plane bundle $\widetilde{\gamma}_{n,3}$ over the oriented Grassmann manifold $\widetilde{G}_{n,3} \cong SO(n)/(SO(3) \times SO(n-3))$ for all n in range $2^t + 2^{t-1} \leq n \leq 2^{t+1} - 4$, $t \geq 3$. Using this knowledge we determine the $\mathbb{Z}_2$-cup-length of $\widetilde{G}_{2^t+2^{t-1}+1,3}$ and $\widetilde{G}_{2^t+2^{t-1}+2,3}$, verifying the corresponding claim of T. Fukaya's conjecture.

1. Introduction and preliminaries

The characteristic rank of a manifold was introduced by Korbaš [5] and later generalized by Naolekar and Thakur [9] to the characteristic rank of a vector bundle defined as follows.

Definition 1.1. Let X be a connected, finite CW-complex and ξ a real vector bundle over X. The *characteristic rank* of the vector bundle ξ, $\mathrm{charrank}(\xi)$, is the greatest integer q, $0 \leq q \leq \dim(X)$, such that every cohomology class in $H^j(X;\mathbb{Z}_2)$ for $0 \leq j \leq q$ can be expressed as a polynomial in the Stiefel–Whitney classes $w_i(\xi)$ of ξ.

In this paper we are mostly interested in the ability to obtain upper bounds for the $\mathbb{Z}_2$-cup-length of a manifold by considering a suitable vector bundle over the manifold and computing its characteristic rank.

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Theorem 1.2 *([9, Theorem 1.2]). Let X be a connected closed smooth d-manifold and let ξ be a vector bundle over X, such that there exists j, $j \leq \operatorname{charrank}(\xi)$, such that every monomial $w_{i_1}(\xi)\dots w_{i_r}(\xi)$ for $0 \leq i_t \leq j$ of degree d is zero. Then*

$$\operatorname{cup}_{\mathbb{Z}_2}(X) \leq 1 + \frac{d-j-1}{r_X}, \tag{1.1}$$

where r_X is the smallest positive integer, such that $\widetilde{H}^{r_X}(X;\mathbb{Z}_2) \neq 0$.

We will be able to utilize Theorem 1.2 to a great degree in the case of the canonical k-plane bundle $\widetilde{\gamma}_{n,k}$ over the oriented Grassmann manifold $\widetilde{G}_{n,k}$ of oriented k-dimensional vector subspaces in $\mathbb{R}^n$.

Since we will be considering only cohomology with $\mathbb{Z}_2$ coefficients, we will abbreviate $H^j(X;\mathbb{Z}_2)$ to $H^j(X)$ and $\operatorname{cup}_{\mathbb{Z}_2}(X)$ to $\operatorname{cup}(X)$ throughout the paper. Also, because of the diffeomorphism $\widetilde{G}_{n,k} \cong \widetilde{G}_{n,n-k}$, we may assume $k \leq n-k$.

For our purposes, the cohomology of the oriented Grassmann manifold $\widetilde{G}_{n,k}$ is best described through its relation to the cohomology of the (unoriented) Grassmann manifold $G_{n,k}$ of k-dimensional vector subspaces in $\mathbb{R}^n$.

The $\mathbb{Z}_2$-cohomology ring of the Grassmann manifold $G_{n,k}$ is (see [1])

$$H^*(G_{n,k}) = \mathbb{Z}_2[w_1, w_2, \dots, w_k]/I_{n,k}, \tag{1.2}$$

where $\dim(w_i) = i$ and the ideal $I_{n,k}$ is generated by k homogeneous polynomials $\bar{w}_{n-k+1}, \bar{w}_{n-k+2}, \dots, \bar{w}_n$, where each $\bar{w}_i$ denotes the i-dimensional component of the formal power series

$$1 + (w_1 + w_2 + \cdots + w_k) + (w_1 + w_2 + \cdots + w_k)^2 + (w_1 + w_2 + \cdots + w_k)^3 + \cdots .$$

Each indeterminate w_i is a representative of the ith Stiefel–Whitney class $w_i(\gamma_{n,k})$ of the canonical k-plane bundle $\gamma_{n,k}$ over $G_{n,k}$.

There is a covering projection $p : \widetilde{G}_{n,k} \to G_{n,k}$, which is universal for $(n,k) \neq (2,1)$. To this 2-fold covering, there is an associated line bundle ξ over $G_{n,k}$, such that $w_1(\xi) = w_1(\gamma_{n,k})$, to which we have Gysin exact sequence [8, Corollary 12.3]

$$\xrightarrow{\psi} H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k}) \xrightarrow{p^*} H^j(\widetilde{G}_{n,k}) \xrightarrow{\psi} H^j(G_{n,k}) \xrightarrow{w_1} \tag{1.3}$$

where $H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k})$ is the homomorphism given by the cup product with the first Stiefel–Whitney class $w_1(\xi) = w_1(\gamma_{n,k})$.

Since the pullback $p^*\gamma_{n,k}$ is isomorphic to $\widetilde{\gamma}_{n,k}$, the ring homomorphism $p^* : H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k})$ (induced by the covering projection $p : \widetilde{G}_{n,k} \to G_{n,k}$) maps each Stiefel–Whitney class $w_i(\gamma_{n,k})$ to $w_i(\widetilde{\gamma}_{n,k})$.

Consequently, the image $\operatorname{Im}(p^* : H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k}))$ is a subspace of the $\mathbb{Z}_2$-vector space $H^j(\widetilde{G}_{n,k})$ consisting only of cohomology classes, which can be expressed as polynomials in the Stiefel–Whitney characteristic classes of $\widetilde{\gamma}_{n,k}$. We shall call it *the characteristic subspace* and denote it $C(j;n,k)$. Moreover (see [10]), the image $\operatorname{Im}(p^*)$ of the ring homomorphism $p^* : H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k})$ is a *self-annihilating* subspace of $H^*(\widetilde{G}_{n,k})$ (that is, for any $x \in C(j;n,k)$ and $y \in C(j';n,k)$ we have $xy = 0$ if $j + j' = k(n-k) = \dim(\widetilde{G}_{n,k})$).

This implies that the characteristic rank of $\widetilde{\gamma}_{n,k}$ is equal to the greatest integer q, such that the homomorphism $p^* : H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k})$ is surjective (that is $H^j(\widetilde{G}_{n,k}) = C(j;n,k)$) for all j, $0 \leq j \leq q$, or, equivalently, that the homomorphism $w_1 : H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$ is injective for all j, $0 \leq j \leq q$.

Hence, in order to compute the characteristic rank of $\widetilde{\gamma}_{n,k}$, it is necessary to study the kernel of $w_1 : H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$. The following is a brief summary of the approach employed in the work of Korbaš and Rusin [7]. Let us denote