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Finitely chainable and totally bounded metric spaces: Equivalent characterizations

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ABSTRACT

A metric space (X, d) is called finitely chainable if for every $\epsilon > 0$, there are finitely many points $p_1, p_2, ..., p_r$ in X and a positive integer m such that every point of X can be joined with some p_j, $1 \leq j \leq r$ by an ϵ-chain of length m. In 1958, Atsuji proved: a metric space (X, d) is finitely chainable if and only if every real-valued uniformly continuous function on (X, d) is bounded. In this paper, we study twenty-five equivalent characterizations of finitely chainable metric spaces, out of which three are entirely new. Here we would like to mention that this study essentially turns the first part of this paper into a sort of an expository research article. A totally bounded metric space is finitely chainable. In order to have a better perception of the difference between total boundedness and finite chainability, several new equivalent characterizations of totally bounded metric spaces are also studied. Moreover, two topological characterizations of metric spaces admitting compatible finitely chainable metrics are given.

1. Introduction

Every real-valued continuous function on a compact space is bounded, but the converse need not be true. In fact, a Tychonoff space X is called pseudocompact if every real-valued continuous function on X is bounded. But for some spaces like metric spaces, compactness and pseudocompactness coincide. In particular, a metric space (X, d) is compact if and only if every real-valued continuous function on (X, d) is bounded. But for a metric space (X, d), one can also consider the family of all real-valued uniformly continuous functions on (X, d) and can make a natural query if there exists any characterization of such

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metric spaces when each such uniformly continuous function is bounded. It can be easily shown that every real-valued uniformly continuous function on a totally bounded metric space (X, d) is bounded. But examples can be found to show that the converse need not be true. That is, a metric space (X, d) on which each real-valued uniformly continuous function is bounded need not be totally bounded, though the metric d has to be necessarily bounded. In order to study metric spaces on which every real-valued uniformly continuous function is bounded, in 1958 Atsuji introduced finitely chainable metric spaces in [4] which were weaker than totally bounded metric spaces but stronger than bounded metric spaces. Atsuji has proved that every real-valued uniformly continuous function on a metric space (X, d) is bounded if and only if (X, d) is finitely chainable. In order to get a notion of boundedness stable under uniformly continuous functions, Hejcman defined these sets in the general setting of uniform spaces and in [14] he called such sets to be bounded. Hejcman has characterized those uniform spaces, on which every real-valued uniformly continuous function is bounded, in terms of these "bounded" sets. Later in [18], Njåstad studied finitely chainable uniform spaces and gave a nice simple characterization of such spaces in terms of covers. It seems that Njåstad was not aware of Hejcman's aforesaid paper [14]. But after Njåstad's works, there has been a gap of nearly four decades during which period, apparently there has been no further work on finitely chainable metric spaces. But then in 2004, O'Farrell characterized finitely chainable metric spaces in terms of ϵ-territory in [10]. Again after a gap of a decade, recently in 2014, Garrido and Meroño have given in [12] two sequential characterizations of finitely chainable metric spaces in terms of Bourbaki–Cauchy and cofinally Bourbaki–Cauchy sequences. But actually, they have called finitely chainable metric spaces Bourbaki-bounded since such metric spaces first appeared in [9]. Subsequently, the term "Bourbaki-bounded" has been used in [1,7,8], where more equivalent conditions for finitely chainable metric spaces have been found in terms of functions.

The main goal of this paper is to present a comprehensive list of equivalent characterizations of finitely chainable metric spaces. From the works of several mathematicians mentioned earlier, we have collected twenty-two such equivalent conditions. In addition, we have given three more new equivalent characterizations of finitely chainable metric spaces in terms of sequences. We study all these equivalent characterizations in the second section. But in order to have easier cycles of proofs, we have split these twenty-five equivalent conditions for a finitely chainable metric space into two results: Theorem 2.15 and Theorem 2.19. But since the split has been decided by us, quite often our proofs have been different from the ones available in the literature. Furthermore, in the second section, we answer the question: under what equivalent conditions, there exists a compatible metric ρ on a metrizable space X such that (X, ρ) becomes finitely chainable.

In order to have a better perception of the difference between total boundedness and finite chainability, one needs to have a much deeper study of total boundedness than available in the literature. In view of this need, we study the equivalent characterizations of totally bounded metric spaces in the third section which serves as the secondary goal of this paper. In Theorem 3.5 and Theorem 3.8, we have given various new equivalent conditions for a metric space to be totally bounded. Also in these two theorems, several equivalent characterizations clearly demarcate the difference between total boundedness and finite chainability. For example, a metric space (X, d) is totally bounded if and only if every locally finite open cover of X having a Lebesgue number has a finite subcover, while (X, d) is finitely chainable if and only if every star finite open cover of X having a Lebesgue number is finite. In fact, Njåstad has proved this result for finitely chainable uniform spaces. We rephrase his result in the context of metric spaces by bringing Lebesgue number into consideration.

The symbols $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Q}$ denote the sets of real numbers, natural numbers and rational numbers respectively. Unless mentioned otherwise, $\mathbb{R}$ and its subsets carry the usual distance metric. If (X, d) is a metric space, $x \in X$ and $\delta > 0$, then $B(x, \delta)$ denotes the open ball in (X, d), centered at x with radius δ. Also, $(\widehat{X}, d)$ denotes the completion of the metric space (X, d).

2. Equivalent characterizations of finitely chainable metric spaces

In this section, we study various characterizations of finitely chainable metric spaces. For that we need to first give a series of definitions and some related propositions.