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The minimum number of Fox colors modulo 13 is 5

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In this article we show that if a knot diagram admits a non-trivial coloring modulo 13 then there is an equivalent diagram which can be colored with 5 colors. Leaning on known results, this implies that the minimum number of colors modulo 13 is 5.

1. Introduction

The Fox colorings of a knot or link ([1], exercise 6 on page 92) are the solutions of a system of linear homogeneous equations read off a diagram of the knot or link at issue. Arcs of the diagram are envisaged as algebraic variables and at each crossing of the diagram the equation "twice the over-arc minus the under-arcs equals zero" is read. The coefficient matrix of this system of equations is called **coloring matrix** of the diagram under study. It has the following feature. Along each row there are exactly one 2, two -1's and the rest of the entries are 0's. It follows that the determinant of the coloring matrix is 0. Furthermore, upon performance of a Reidemeister move on the diagram, the coloring matrix corresponding to the new diagram relates to the coloring matrix corresponding to the former diagram by elementary transformations on matrices. Thus the equivalence class of the coloring matrix for any diagram of the knot under study is an invariant. Let us choose for representative of this equivalence class the Smith Normal Form (SNF) of

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the coloring matrix. Since the determinant of these matrices is 0, then one of the entries of the diagonal of the SNF is 0 and this corresponds to the monochromatic solutions i.e., the solutions obtained by assigning the same color (number) to each arc of the diagram. Polichromatic solutions, also known as non-trivial solutions, are obtained if there is at least one more 0 along the diagonal of the SNF. Especially in the case of knots, this involves working over the modular integers for a specific prime modulus p. If our knot or link, L, admits non-trivial colorings over a modulus p, Harary and Kauffman, [2], introduced the minimum number of colors of L mod p, notation $mincol_p(L)$. This is the minimum number of distinct colors it takes to assemble a non-trivial coloring mod p, the minimum being taken over all diagrams of the link at issue.

At this point we warn the reader that any knot or link considered in this article has non-zero determinant, the **determinant** of the knot or link being the product of the entries of the diagonal of its SNF but the 0 referred to above. As a matter of fact, a link with zero determinant is colorable modulo any prime which makes these links quite special and deserving a separate article.

There is a number of articles on the topic of minimum number of colors [3,4,6–11,13–15]. In [15], Satoh developed a technique for finding the minimum number of colors over a fixed modulus but on an otherwise arbitrary situation. One considers a diagram equipped with a non-trivial coloring on the given modulus and using all available colors. The idea is then to remove one color at a time until one cannot remove any more colors. To remove one color one assumes it shows up in the diagram in all possible ways. Specifically, we assume it shows up as the color of a monochromatic crossing and devise a procedure to eliminate that color from this monochromatic crossing; we repeat the procedure for each monochromatic crossing bearing this color. These will be called the α instances. Then we assume the color at stake shows up at the over-arc of a polychromatic crossing and devise a procedure to eliminate it from the over-arc; and we repeat the procedure for all polychromatic crossings with this color on the over-arc. We call these the β instances. Finally we assume the color shows up on an under-arc connecting two crossings, and devise a procedure to eliminate it and repeat it over all such situations. Here we distinguish two cases. If the adjacent over-arcs bear distinct colors we call them γ instances; otherwise δ instances. In each of the α, β, γ, and δ instances, the procedure for eliminating the color consists of performing Reidemeister moves (accompanied by the unique rearrangement of colors that yields a coloring in the new diagram) so that the color at issue is eliminated. The transformations or sequences of Reidemeister moves that take care of α instances will be called α_i's and analogously for the other instances. If color c has been successfully eliminated in each of the α, β, γ, and δ instances, one moves on to color c' and iterates the procedure taking into consideration this time that color c is no longer there (nor the colors previously removed). Satoh applied this technique successfully in [15] to show that mod 5 four colors suffice. Then Oshiro [13] made the first impressive use of this technique by eliminating a string of three colors mod 7 thus showing that mod 7, four colors suffice. Using the same technique, Cheng et al. [3] proved that at most six colors are needed mod 11, and Nakamura et al. [12] proved further that five is the minimum number of colors for any knot or link admitting non-trivial 11-colorings. In the current article we apply Satoh's technique to prove the following result.

Theorem 1.1. *For any knot or link (with non-zero determinant) admitting non-trivial colorings modulo* 13, *there is a diagram of it equipped with a non-trivial coloring modulo* 13 *using five colors.*

Corollary 1.1. *If L is a knot or link in the conditions of Theorem 1.1, then*

$$mincol_{13}\, L = 5.$$

Furthermore, since there is essentially one set of five colors modulo 13 *which can color a non-trivial coloring, there is a Universal* 13*-Minimal Sufficient Set of Colors, in the sense of [10].*

Proof. Since it is known [11,9,6] that the minimum number of colors modulo 13 has to be at least 5 then the equality follows from the Theorem.