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Monadic second-order properties of very sparse random graphs ☆

L.B. Ostrovsky, M.E. Zhukovskii *

Moscow Institute of Physics and Technology, Laboratory of Advanced Combinatorics and Network Applications, Russian Federation

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ABSTRACT

We study asymptotical probabilities of first order and monadic second order properties of Bernoulli random graph $G(n, n^{-a})$. The random graph obeys FO (MSO) zero-one k-law (k is a positive integer) if, for any first order (monadic second order) formulae with quantifier depth at most k, it is true for $G(n, n^{-a})$ with probability tending to 0 or to 1. Zero-one k-laws are well studied only for the first order language and $a < 1$. We obtain new zero-one k-laws (both for first order and monadic second order languages) when $a > 1$. Proofs of these results are based on the earlier studies of first order equivalence classes and our study of monadic second order equivalence classes. The respective results are of interest by themselves.

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1. Logic of the random graph

In 1959, P. Erdős and A. Rényi, and independently E. Gilbert, introduced two closely related random graph models. While a seminal paper of Erdős and Rényi [7] gave birth to rapidly developing area of mentioned random graphs, the more popular Bernoulli model $G(n, p)$ was proposed by Gilbert in [8] (but actually this model is much older). In the paper, we study asymptotical probabilities of first order and monadic second order properties of the random graph $G(n, p)$ (see also [3,11,15,21]). Recall that edges in this graph on the set of vertices $V_n = \{1, \ldots, n\}$ appear independently with probability p (i.e., for each undirected graph $H = (V_n, E)$ without loops and multiple edges, the equality $\mathsf{P}(G(n, p) = H) = p^{|E|}(1-p)^{\binom{n}{2}-|E|}$ holds).

Formulae in the first order language of graphs (first order formulae) [1,5,21,17,15] are constructed using relational symbols $\sim$ (the symbol of adjacency) and $=$; logical connectivities $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\vee$, $\wedge$; variables

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* Corresponding author.
E-mail address: zhukmax@gmail.com (M.E. Zhukovskii).

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$x, y, x_1, \ldots$ (that express vertices of a graph); and quantifiers $\forall$, $\exists$. Monadic second order formulae [9,16] are built of the above symbols of the first order language and variables $X, Y, X_1, \ldots$ that express unary predicates. Following [1,5,21,17], we call a number of nested quantifiers in the longest chain of nested quantifiers of a formula ϕ *the quantifier depth* $q(\phi)$. For example, the formula

$$(\forall X\ ([\exists x_1 \exists x_2\ (X(x_1) \wedge (\neg(X(x_2))))] \Rightarrow [\exists y \exists z\ (X(y) \wedge (\neg(X(z))) \wedge (y \sim z))]))$$

has quantifier depth 3 and expresses the property of being connected. It is known that this property is not expressed by a first order formula (to the best of our knowledge, the first proof of this fact appears in [10] for sentences with quantifier depth 4; the general proof can be found in, e.g., [21]).

We say that $G(n,p)$ *obeys FO zero-one law* (*MSO zero-one law*) if, for any first order formula (monadic second order formula), it is either true asymptotically almost surely (a.a.s.) or false a.a.s. (as $n \to \infty$). Note that we use the phrase "asymptotically almost surely" when the considered property of the random graph holds with asymptotical probability 1 (as $n \to \infty$). In some papers on random graphs, the phrase "almost surely" (a.s.) is used instead. However, in probability theory the latter phrase normally means that the considered event holds with probability 1 (this is why we choose the first phrase).

In 1988, S. Shelah and J. Spencer [14] proved the following zero-one law for the random graph $G(n, n^{-\alpha})$.

Theorem 1. *Let $\alpha > 0$. The random graph $G(n, n^{-\alpha})$ does not obey FO zero-one law if and only if either $\alpha \in (0,1] \cap \mathbb{Q}$ or $\alpha = 1 + 1/l$ for some integer l.*

Obviously, there is no MSO zero-one law when even FO zero-one law does not hold. In 1993, J. Tyszkiewicz [16] proved that $G(n, n^{-\alpha})$ does not obey MSO zero-one law for irrational $\alpha \in (0,1)$ also. When $\alpha > 1$ and does not equal to $1 + 1/l$ for any positive integer l, MSO zero-one law holds. The last statement simply follows from standard arguments from the theory of logical equivalence. We believe this result is known. Unfortunately, we did not find it in the related papers. So we give the proof in Section 4.2. Below, we state the general result on MSO zero-one law for $G(n, n^{-\alpha})$.

Theorem 2. *Let $\alpha > 0$. The random graph $G(n, n^{-\alpha})$ does not obey MSO zero-one law if and only if either $\alpha \in (0,1]$ or $\alpha = 1 + 1/l$ for some integer l.*

For a formula ϕ consider the set $S(\phi)$ of α such that $G(n, n^{-\alpha})$ does not obey the zero-one law for the fixed formula ϕ. Neither theorem gives an explanation of how the set $S(\phi)$ depends on ϕ (or even on a quantifier depth of this formula). However, better insight into an asymptotical behavior of probabilities of the properties expressed by first order and monadic second order formulae is given by zero-one k-laws (k is a positive integer upper bound for quantifier depths of formulae, see Section 3), which are well studied only for the first order language and $\alpha \leq 1$ (see, e.g., [21]). In this paper, we obtain new zero-one k-laws (both for first order and monadic second order languages) when $\alpha > 1$ and give their statements in Section 3. Proofs of these results are based on the earlier study of first order equivalence classes and our study of monadic second order equivalence classes (see Section 2). The respective results are of interest by themselves.

2. Logical equivalence

For two graphs G and H and any positive integer k, the notation $G \equiv_k^{\text{FO; graphs}} H$ denotes that any first order formula ϕ with $q(\phi) \leq k$ is true on both G and H or false on both G and H. The notation $G \equiv_k^{\text{MSO; graphs}} H$ is defined similarly. Obviously, $\equiv_k^{\text{FO; graphs}}$ and $\equiv_k^{\text{MSO; graphs}}$ are both equivalence relations on the set of all graphs. Moreover, for every k there are only finitely many equivalence classes (see, e.g., [5]) and an upper bound for the cardinality $r_k^{\text{FO; graphs}}$ of the set of all $\equiv_k^{\text{FO; graphs}}$-equivalence classes $\mathcal{R}_k^{\text{FO; graphs}}$

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