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Full-splitting Miller trees and infinitely often equal reals

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ABSTRACT

We investigate two closely related partial orders of trees on ω^ω: the *full-splitting Miller trees* and the *infinitely often equal trees*, as well as their corresponding σ-ideals. The former notion was considered by Newelski and Rosłanowski while the latter involves a correction of a result of Spinas. We consider some Marczewski-style regularity properties based on these trees, which turn out to be closely related to the property of Baire, and look at the dichotomies of Newelski–Rosłanowski and Spinas for higher projective pointclasses. We also provide some insight concerning a question of Fremlin whether one can add an infinitely often equal real without adding a Cohen real, which was recently solved by Zapletal.

1. Introduction

A common theme in descriptive set theory and forcing theory on the reals are perfect-set-style *dichotomy theorems*—statements asserting that all Borel (or analytic) sets are either in a σ-ideal $\mathfrak{I}$ on ω^ω, or else contain the branches of a certain kind of tree. When $\mathbb{P}$ denotes the partial order of these trees ordered by inclusion, such a theorem guarantees that there is a dense embedding

$$\mathbb{P} \hookrightarrow_d \mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$$

from $\mathbb{P}$ to the partial order of Borel sets positive with respect to $\mathfrak{I}$ (also ordered by inclusion), and hence that the two posets are forcing-equivalent. The most famous result of this kind is the original *perfect set theorem*, showing that the Sacks partial order (perfect trees ordered by inclusion) densely embeds into the partial order of uncountable Borel sets. Jindřich Zapletal [18,19] developed an extensive theory of *idealized forcing*,

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i.e., forcing with $\mathcal{B}(\omega^\omega) \setminus \mathfrak{I}$ for various σ-ideals $\mathfrak{I}$ on the reals. In Zapletal's framework, properties of the forcing can be studied directly using properties of the σ-ideal. On the other hand, there is a long-established tradition of studying forcing properties using combinatorics on trees. A dichotomy theorem provides the best of both worlds, since it allows us to freely switch back and forth between the "idealized" and the "tree" framework, depending on which suits the situation better.

In this paper we consider two closely related dichotomies. The following two definitions are due to Newelski and Rosłanowski [14].

Definition 1.1. A tree $T \subseteq \omega^{<\omega}$ is called a *full-splitting Miller tree* iff every $t \in T$ has an extension $s \in T$ such that s is *full-splitting*, i.e., $s^\frown \langle n \rangle \in T$ for every n. Let $\mathbb{FM}$ denote the partial order of full-splitting Miller trees ordered by inclusion.

Definition 1.2. For $f : \omega^{<\omega} \to \omega$, let

$$D_f := \{x \in \omega^\omega \mid \forall^\infty n\, (x(n) \neq f(x{\restriction}n))\}.$$

Then $\mathfrak{D}_\omega := \{A \subseteq \omega^\omega \mid A \subseteq D_f \text{ for some } f\}$.

The original motivation of [14] was the connection to infinite games of the same kind as used by Morton Davis in [4] in the proof of the perfect set theorem from determinacy, but played on ω^ω instead of 2^ω. Let $G^*(A)$ be the game in which Player I chooses $s_i \in \omega^{<\omega} \setminus \{\varnothing\}$ and Player II chooses $n_i \in \omega$, and I wins iff $s_0{}^\frown \langle n_0 \rangle{}^\frown s_1{}^\frown \langle n_1 \rangle{}^\frown \cdots \in A$. It is easy to see (cf. [15]) that Player I wins $G^*(A)$ if and only if there exists a tree $T \in \mathbb{FM}$ such that $[T] \subseteq A$, and Player II wins $G^*(A)$ if and only if $A \in \mathfrak{D}_\omega$. General properties of so-called *Mycielski ideals* (i.e., ideals of sets for which II wins a corresponding game) imply that $\mathfrak{D}_\omega$ is a σ-ideal on ω^ω. Using Solovay's "unfolding" method (see e.g. [9, Exercise 27.14]) it follows from the determinacy of closed games that analytic sets are either $\mathfrak{D}_\omega$-small or contain $[T]$ for some $T \in \mathbb{FM}$.

The next concept is due to Spinas [16].

Definition 1.3. For every $x \in \omega^\omega$ let $K_x := \{y \in \omega^\omega \mid \forall^\infty n\ (x(n) \neq y(n))\}$, and let $\mathfrak{I}_{\text{ioe}}$ be the σ-ideal generated by K_x, for $x \in \omega^\omega$.

In [16], $\mathfrak{I}_{\text{ioe}}$-positive sets were called "countably infinitely often equal families", since a set A is $\mathfrak{I}_{\text{ioe}}$-positive if and only if for every countable sequence of reals $\{x_i \mid i < \omega\}$ there exists $a \in A$ which hits every x_i infinitely often. The following result was claimed in [16, Theorem 3.3]: "every analytic set is either $\mathfrak{I}_{\text{ioe}}$-small or contains $[T]$ for some $T \in \mathbb{FM}$". This dichotomy is clearly in error, as the simple example below shows:

Example 1.4. Let T be the tree on $\omega^{<\omega}$ defined as follows:

- If $|s|$ is even then $\text{succ}_T(s) = \{0, 1\}$.
- If $|s|$ is odd then $\text{succ}_T(s) = \begin{cases} 2\mathbb{N} & \text{if } s(|s|-1) = 0 \\ 2\mathbb{N}+1 & \text{if } s(|s|-1) = 1 \end{cases}$

where $\text{succ}_T(s) := \{n \mid s^\frown \langle n \rangle \in T\}$. Clearly T is $\mathfrak{I}_{\text{ioe}}$-positive but cannot contain a full-splitting subtree.

The correct dichotomy for the ideal $\mathfrak{I}_{\text{ioe}}$ involves a subtle modification of the concept of a full-splitting tree, suggested by Spinas in private communication.

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